FERMAT'S LOST PROOF AND GREEK MATHEMATICS: THEIR HISTORY AND PHILOSOPHY

Fermat's Last Theorem (FLT 1637) states without proof that if a>b>c>0 are integers and p>2 a prime, then [a to the power p] *is not equal to* [b to the power p + c to the power p]. Its negation, the Odd Prime (Diophantine) Equation, in turn, is the usually tried *Counter-hypothesis* in indirect proofs to FLT (1637). It suffices to prove FLT (1637) for {a,b,c} pairwise relatively prime integers, and for the exponent 4 and for all odd primes p without loss of generality (cf. RIBENBOIM 1979, pp. 35-36). As to the exponent 4, Fermat's proof to the Biquadratic Equation is preserved. It serves as the standard of comparison for structure and rigour in our *historical reconstruction* of Fermat's lost proof to FLT (1637).

Our target is, first, the reconstruction of Fermat's *heuristics*. We demonstrate that his method was based on Pappus' description of the ancient Greek method of *Analysis-and-Synthesis*. Its «theoretical variant» is a *method of proof* and a particular case of *reductio ad absurdum*. Our second task is to show that the proof is valid today, too, although Pappus' method differs from the canons of modern logic. We employ Fermat's *Little Theorem* (FT, stated 1640 without proof) to an *Auxiliary Construction* **{H}** that accomplishes the task: we establish an algorithm that yields an inductive proof to FT (1640). The first step of the algorithm, in turn, gives the decisive point in the proof to FLT (1640) by a few propositions of Euclid's *Elements*. The algorithm proceeds in the order of *prime numbers*, not in the order of the natural numbers and, therefore, Eratosthenes' κόσκινον and the Fundamental Theorem of Arithmetic are necessary (just as in the modern «Sieve methods» of today's prime number searches). Quite literally, then, Fermat's proof reconstructed, seen in the historical perspective, is an heir to ancient Greek mathematics. It is worth Fermat's genius and without loss of generality. And it is decisive, being ultimately anchored to the distinction of odd and even integers, the Fundamental Theorem of Arithmetic, and the universal FT (1640).

Yet, in an enterprise concerned with the history and philosophy of mathematics *and* heuristics, criticism is to be expected from all angles. Is the historical evidence sufficient? Is the reconstructed proof valid even in our days, or is it a proof at all? How can we understand Fermat's heuristics and claim of «the most remarkable proof» (1637), as his only answer to contemporary critics was the statement of his Little Theorem (1640)? Can a genius of his calibre be analysed in rational terms?

An historical reconstruction must employ methods known to Fermat according to his *Oeuvres*. It must also avoid anachronisms, such as methods and principles invented by later mathematicians (e.g. Mathematical Induction and the well-ordering of the natural numbers). In particular, our historical reconstruction is entirely independent of Andrew Wiles' modern proof, the feat accomplished in 1994. No one can seriously conjecture that Fermat knew in 1637 anything of the modularity of elliptic functions or other modern results, in spite of the fact that he anticipated new ideas from optics to the calculus of probability. Nevertheless, two entirely different proofs to the same problem, separated by 373 years, is a treasure for the philosophy of mathematics.

Furthermore, it is good to know the historical parallels and eventual predecessors. We must have a clear picture of Fermat's options and alternative proofs, as well as of the blind alleys he probably tried. Fortunately, there is a reasonable explanation why the Little Theorem (FT 1640) can be employed without anachronism. *Hardy& Wright* note on (Euler's) proof that it «depends on the simplest arithmetical properties of the binomial coefficients». They are known from Pascal's Triangle and Fermat was in correspondence with Pascal, who 1640 distinguished himself with the conic sections at the age of 16. After that year Fermat never mentioned his «truly remarkable proof». Presumably FT (1640) was Fermat's answer to his critics. *Hardy & Wright's* note is not concerned with him. We submit that the proofs to FT (1640) and FLT (1637) are intertwined and due to Fermat alone, not partly to Pascal. Our reconstruction of Fermat's heuristics and proof is based on three papers (Note 1). For the general orientation of FLT (1637), cf. n. 2.

A fresh angle to the evolution of proofs is offered by Gian-Carlo Rota, of MIT, in his recent book *Indiscrete Thoughts*. Amid mathematical anecdotes and caricatures of Great Men, he outlines a practising mathematician's view on new, ever improving proofs. His yardstick is the diminishing number of pages. The fresh point is a *sine qua non* revolution in conceptual analysis, as the prerequisite for the improvement of proof. This train of thought can be extended so as to call a *proof definitive or decisive*, insofar as it is conducted in the realm of the most fundamental and compelling logical means, terms and distinctions.

When such a decisive proof is met in Euclid, if the concatenation of logical steps leads to the distinction of «odd» and «even», say, we often call the proof beautiful. Likewise, if we can refer the proof to the Fundamental Theory of Arithmetic, say, it is a good reason to call it definitive. As it happens, our historical reconstruction meets these expectations. Therefore, it also satisfies Fermat's highest qualitative criterion of a proof, expressed in his words: «I have discovered a truly remarkable proof...». That criterion is anchored to history and contemporary state of the art and justified in terms of the era's logic and mathematics. We submit that its generality is still valid.

Some steps in our reconstruction of Fermat's heuristics may not comply with the routines of modern logic. But they comply with Pappus' account of the ancient Greek method of Analysis-and-Synthesis in its theoretical variant, a *method of proof*. The final outcome is a proof to FT (1640), embedded in the proof to FLT (1637). We submit this is Fermat's «truly remarkable proof». It holds even today, for the main traits of proof are still the same: (i) the principal remainder of a formula on division by a prime is obtained by adding the principal remainders of its parts, (ii) the factorisation into primes of a number is unique, (iii) FT (1640) is a universal theorem for all primes from 2 upward, (iv) generality is the goal of a proof: there is no loss of generality if the two variants of FT (1640) *and* the four formulae ensuing from their application are equivalent, and (v) any two formulae of the same parity can be juxtaposed and simplified relaying on the parity relation's reflexiveness, symmetry and transitivity. These pivotal traits of mathematical proof persist.

Fermat had several openings for a proof, but only one passed his most stringent qualitative criterion.

Aiming at a decisive *historical reconstruction* of FLT (1637), we have adopted a three-part strategy, developed in my 1985 paper «Proof, History and Heuristics» (cf. n. 3). A full-blooded mathematician may depreciate the history, and hide his heuristics like Fermat. We feel that historical parallels and even alluring mistakes make the feat more glorious. If we enter the same blind alleys, we appreciate the intuition of the Prince of Amateurs. These three aspects are all needed in *problem-solving*, of which I gave examples. I also stated that if FLT (1637) fails and its negation holds, then the triple {a,b,c} represents an acute-angled triangle. That fact later made the backbone of our historical reconstruction of Fermat's *heuristics*. But in the search for the *proof*, we finally set the *Counter-hypothesis* at stake, too, discovering a singularity.

In this *introductory essay*, I shall focus on the auxiliary constructions and proofs from the point of view of the history and philosophy of mathematics. The formal proofs, as well as further heuristic and historical discussions, are included in our unpublished paper: E. Maula, E. Kasanen, J. Mattila, Fermat's Marginal Note and its Greek Historical Proof.

On Auxiliary Drawings and Constructions

In our paper we demonstrate that Fermat's «Infinite (or Indefinite) Descent» is a special case of *reductio ad absurdum*, derived from the ancient Greek method of Analysis-and-Synthesis. Fermat, who studied Greek poetry, had learned the method from Pappus' description. This method of proof differs from the canons of modern proof. It is easy to glean from Pappus, a competent mathematician of the third century, more than he really yields. Just that some modern scholars have done (cf. n. 4). And the same holds for Fermat: he is credited with later results.

In our first Fermat-paper (1989) we proved that, if FLT (1637) fails and its negation holds, then the triples {a,b,c} to all powers up to the prime (p), represent scalene triangles, both acute-angled and obtuse-angled, but never right-angled triangles. This is in our 2010 paper the *heuristic Prop. 1.*

Rewriting a = (a-c) + (b+c-a) + (a-b), b = (a-c) + (b+c-a) and c = (a-b) + (b+c-a), we have the three identities of the *Auxiliary Drawing* of the scalene, acute-angled triangle {a,b,c}. The right-hand sides of the identies lend themselves to what we call the *characteristics* of scalene triangles. They distinguish acute triangles from the obtuse ones by means of implication, not by inequality as in Heron's *definitions*. Nevertheless, these characteristics admit of higher powers, which is not possible when we ascend from the triangle {a,b,c} up to the prime p:th powers of its sides. This is, of course, the main formal obstacle in the mainline mathematical solutions to FLT (1637). It has made the progress slow but, demanding particular ingenuity, has greatly advanced number theory also. Our Auxiliary Drawing alone cannot solve the problem. Nor can the divisibility rules for polynomials do so, yet they must be maintained as constraints **{C}** necessary for any (eventual) solution.

Is it worthwhile to repeat the elementary approaches, certainly tried time and again? In our 1989 paper, we experimented with other ideas, for instance giving up the sufficient condition that a>b>c>0 are pairwise relatively prime. Thus we studied the form 2p|(b+c-a) vis-à-vis the then recent results of the mainline mathematics. To our surprise, this form worked better than Gruenert's (1856) lower bound for an eventual solution, and Perisastri's (1969) bound z < square x is less stringent than ours for an acute-angled triangle {z,y,x} where z>y>x>0. Likewise, our result in the first Fermat-paper (*Prop. 4*) is more general than Terjanian's (1977) result on even exponents. Our experimental form is obtained if the *Counter-hypothesis* holds but p|a, p|b and p|c, that is, a>b>c>0 are *not* pairwise relatively prime. As this results from the use of FT (1640), there must be something wrong with the negation of FLT (1637). But what is it? Our heuristic experiment reveals that we do not know enough about the acute triangle {a,b,c} and that FT (1640) might be a means to extract that knowledge. It presupposes, however, that we find an appropriate Auxiliary Construction for it. We must dispose of the experimental form if a definitive solution to FLT (1637) will be found.

After more experiments («everybody knows that mathematics is an experimental discipline!») and comparisons between mathematical induction

(Prof. Mattila), the arithmetic of the inequals (Prof. Kasanen), and the method of Pappus (myself) applied to the proof of an instructive lemma set forth in our second paper (2004), we concluded that the fuzzy logic does not corroborate our historical reconstruction. Nor could we create a general theory of the triangular numbers, *i.e.* the arithmetic of integer triples, which had been searched for in vain by Hamilton.

On the positive side, it dawned on us at last that Fermat's lost proof must be based on facts that we already know. That is, by *Prop. 1,* {a,b,c} is an acute-angled triangle with integer sides a>b>c>0 pairwise relatively prime. Hence Fermat could employ *both* arithmetical *and* geometrical methods to one and the same problem, just as in the Biquadratic Equation. Accordingly, that two-pronged method should direct us in our search for an Auxiliary Construction to conduct the proof.

In 2010 we finally found it, a perfect match for FT (1640). We call it **{H}**. It is an edifice where the rising powers of {a,b,c} in the form {b+c-a}, {the square of b + the square of c – the square of a}, {the cube of b + the cube of c – the cube of a}, and so forth, are set on top of each other in the order of the exponents. It has the advantage that we need not worry about the transit from one form to the next in the rising powers of {a,b,c}. The edifice is an auxiliary invention, not an outcome of algebraic or arithmetical transformation of one triple to another. It also resembles the negation of FLT (1637) on its uppermost rung (when the exponent is *p*). Yet it is founded on geometrical facts. According to the heuristictic *Prop. 1,* every triple represents a scalene triangle, either acute-angled or obtuse-angled, and for every scalene triangle there is the smallest Heronic constituent. We omit the square root, coefficients, and the other constituents {(a+b+c), (a-b+c), (a+b-c)} in Heron's Formula. **{H}** is not logically dependent on the *Counter-hypothesis,* or on *Prop. 1.*

Now we apply FT (1640) to inquire into the divisibility by primes (from 2 up to *p*), of the pertinent constituent. That will provide us with information about the corresponding triangles. Had we proven with Fermat, first the FT (1640), we should have seen it immediately that FT (1640) and **{H}** are *complementary concepts.* They combine arithmetical operations with geometrical facts in the same way as in Fermat's proof to the Biquadratic Equation. **{H}** is to be conceived of as a touchingstone for divisibility by primes on the set of integers {a,b,c} pairwise relatively prime, where a>b>c>0, and on their rising powers. This is the right tool to the required task. We must note that without FT (1640), we know next to nothing about the scalene triangles obtained by *Prop. 1* either. On the other hand, FT (1640) is the *only* tool discovered from Fermat's kit that is able to glean the needed information. It concerns the divisibility of formulae by primes, beginning with parity on the lowest rung.

On the Proofs

What has made it difficult for modern mathematicians to understand Fermat is our era's all too common departmentalization. A good example is in *Hardy&Wright*'s rendering of Fermat's proof to the Biquadratic Equation. In their elegant proof, Fermat's geometrical predilection is forgotten, and in their assumption of the smallest value for a term, pertinent geometrical facts are skipped. It is unfair to stump Fermat's algorithmic method of Infinite (or Indefinite) Descent through the *akoloutha* in a fixed order, *i.e.* a concatenation of logical steps, which he describes in detail, into a single step in modern proof. In his own time, Fermat had support from Scholastic dichotomies like the interwined *essentia* and *existentia* or the Platonic Forms and their exemplifications, later diluted in literature theory into *form and content*. In Fermat's thought, the dichotomy is accentuated and made use of. We would like to say that Fermat anticipated Frege's much later distinction of *Sinn* and *Beteutung*. Be that as it may, Fermat did not hesitate to divide by primes the triples in **{H}**, which represented, or at least referred to, scalene triangles with integer sides.

These divisibilities are investigated by means of FT (1640), which distinguishes between two cases. If (n) is an integer and (p) a prime such that $p|n$, then $p|$ [(n to the power p) – n], and if (p) does not divide (n), then $p|$[(n to the power p-1) – 1]. Fermat stated (*Oeuvres* ii, 209) but did not give a proof to this universal principle, which extends to all primes (from 2, 3, and 5 upwards), in the present case up to the prime p. Its first proof in 1736 and its generalization in 1760 are due to Euler.

The proof to Fermat's Last Theorem is the hub of our endeavour in *Prop. 4* in our third paper (2010). We no more assume that the *Counter-hypothesis* holds, but demonstrate in it a self-contradictory trait. The means to do so is the application of FT (1640) to {square b + square c – square a} on the second rung of **{H}**, where the variants of FT are *equivalent* and, resulting from the application, the deformed Heronic constituents are all even. Besides, they also are *equivalent* on the same condition as the variants of FT: (a,b,c) are all *odd*.

Descending the prime number rungs of **{H}**, toward 7, 5, 3 and 2, we always repeat the *same steps of proof*. In modern terms, this is an algorithm. It is a «Prime Number Induction» that gives the driving force to FT (1640) and connects the κόσκινον with the Auxiliary Drawing.

It is sufficient to demonstrate the operation while considering whether 2 divides (the square of b + the square of c – the square of a). Suppose first that 2 does *not divide* {a, b, or c}. Thus 2|[b+c-a-1], by FT (1640). Hence 2 cannot divide [b+c-a], for these two numbers are consecutive and prime to one another. For the relative primality, cf. *Elem. vii. 1, 21-32* and *ix. 14-17*. In view of the relative primality of the equipotent powers of {a,b,c} on the prime number rungs of **{H}**, *Elem. v. Prop. 27* is central. By the Fundamental Theorem of Arithmetic, the factorisation into primes is unique. In everyday terms, [b+c-a] is odd and [b+c-a-1] is even. This is the homeyard of Fermat, the arithmetic of parities was learned from *Elem. ix. 21-31; ix. 29* warrants that if {a,b,c} are all odds, their *p*:th powers are all odd, which is a pivotal feature of the proof.

We next ask what happens if either 2|a, or 2|b, or 2|c, and apply FT (1640) to (the square of b + the square of c – the square of a). When FT (1640) is being applied, in its two variants, the outcome is seven deformed Heronic constituents, which are all *even*. None of these constituents is defined on the value odd. But the *even* forms {(a+b+c) – square of (a) -2}, {square of (b) – (a+b-c)}, {square of (c) – (a-b+c)} occur on two lines each, including the last line, while the *even* form {b+c-a-1} occurs on the last line alone. We may and must compare these even forms with one another (cf. Hardy and Wright, *theorems 70, 71*). It is proved (in our paper, *Prop. 4*) that a contradiction or an absurdity ensues in three cases, while on the last line when (2) does *not divide* (abc), the two variants of FT (1640) are equal and the four deformed Heronic constituents are equal without loss to generality. Here, finally, we meet the pivotal *reductio ad absurdum*. Due to the architecture of the ancient Greek method of Analysis-and-Synthesis, it occurs on the last rung of the Auxiliary Drawing, together with the only even prime 2. That is to say, all alternatives are eliminated by contradiction, and there is no contradiction when all members of the triple {a,b,c} are *odd*. The *reductio ad absurdum* did not emerge from the *Counter-hypothesis* as one might have expected.

This is a conclusion that may differ from modern canons of proof, and we may call it heuristics. But there is more than meets the modern eye. *In statu nascendi*, it is the proof of FT (1640) inside the proof to FLT (1637) by means of a «Prime Number Induction»: a strange but valid proof. The difference sits in the simultaneity of the two proofs. We may recall that, in contradistinction to modern mathematicians, Fermat had to conduct them both. He had no ready proposition to begin with. He invented both two. In modern terms, his secret is *simultaneity* of different angles combined with deep knowledge of his Greek predecessors, and his benchmark is the *generality* of the proofs.

The odd parity and relative primality of {a,b,c} is inherited in the higher powers in **{H}**, by *Elem. vii. 27* and *ix. 29*. Hence also the *p*:th powers of {a,b,c} are all odd. Thus the sum of the *p*:th powers of {b} and {c} is even, but the *p*:th power of {a} is odd. As they are of different parity, they cannot be equal.

This is «the truly remarkable proof» to FLT (1637), the heuristics of which we have historically reconstructed of the elements and methods known to

Fermat. Owing to a self-contradictory trait, the *Counter-hypothesis*, the negation of FLT (1637), annihilates itself and thus (the solution to the Biquadratic Equation corroborating), FLT (1637) is proved. We stand by the sufficient condition, that {a,b,c} are pairwise relatively prime. Hence the intriguing formula 2p|(b+c-a), from our first paper, can be firmly set aside. The path to the proof is compatible with Fermat's verbal description of his method, in particular with its earlier stage used to prove negative statements on primes. It is also in full agreement, as regards the structure of the proof, with the Biquadratic Equation. This is shown in a detailed comparison in our unpublished paper. And as for the historical parallels, I have discussed the vital question whether {a,b,c} are all odd, also by the technique of congruences. Note the historical fact, however, that congruences became operational in Gauss' time, two centuries later.

But it is not the end of the matter. One may still have the impression that we have skipped some difficulties while jumping from one prime number rung to the next in **{H}**. Not by chance, however, we meet an old friend here: Eratosthenes' κόσκινον.

We begin the Infinite (or Indefinite) Ascent from {b+c-a}, step by step toward the highest prime number rung. On each rung we prove, in the same way always, that neither the appropriate Heronic constituent is divisible by the prime number in question, nor the equipotent powers of {a,b,c} separately are divisible by it. That discussion is given in an *Appendix* to our third paper. As «any number either is prime or is measured by some prime number», *Elem. vii. 32*, we have discussed, in our heuristical reconstruction, the possibility that if {a,b,c} are all odd, they are also *primes* greater than (*p*), or composite consisting of factors greater than *p*. This result, by the same means as the «Sieve» operates, covers also the composite number rungs of **{H}** and the Heronic constituents on them. This result is not needed in our historical reconstruction of FLT (1637). However, it makes sure that we have uncovered Fermat's method of Infinite (or Indefinite) Descent.

In modern terms, we would like to dub it the *Prime Number Induction*. An inductive method it is, but it differs from the mathematical induction. The kernel of their difference is the principle of the well-ordering of the natural numbers. Fermat never states it and never makes use of it, albeit modern mathematicians often presume that he knew it, or else that it is embedded in his method. That anachronism is obvious when we note that neither the triples nor the scalene triangles {a,b,c} can be so ordered; they must be ordered according to the sizes of the triangles (as Fermat often does elsewhere in general terms «greater» and «smaller»), or in the order of primes. This form of induction is actually more interesting than mathematical induction, because FT (1640), which is an essential part of it, reduces the number of principal remainders on division by a prime into two.

The Auxiliary Construction **{H}** is as permissible as any Auxiliary Drawings, of course, but in itself it does not bring about any mathematical novelty. From an architect's point of view, it may be conceived of as three geometrical series, well known from *Elem. ix. 12-13*, and their sums could be counted as well (*Elem. ix. 35*).

If we do not study history, we cannot predict the new ideas developing in the past any the more than old things falling apart in the future. As an example, we recall Hilbert's predictions in 1900. They went strong almost for one century. The case is somewhat different if we happen to uncover a mathematical method, omitted or forgotten in the mainstream mathematics. We must try it to other problems of course. In this case, we must compare our historical reconstruction with Andrew Wiles' proof to FLT (1637). In the Prime Number Induction we also have a means to tackle Hamilton's problem: Why integer triples have no arithmetic, although ordered pairs and quaternions have?

But what else could we predict or offer? Distant deities, faraway Muses: Clio, Urania, Euterpe and your six sisters (one for each child of mine), and the silent Love, the source of inspiration behind our trilogy: I invoke you all. Tell me, what is the path from Fermat to us, off the mainstream of mathematics? How can we uncover other secrets and solve other enigmas?

From Fermat to Goldbach

In our work "Fermat's Marginal Note" we prove first that if the *Counter-hypothesis* (the negation of FLT 1637) holds for the pairwise relatively prime integers a>b>c>0 and for the prime exponent *p* equal to or greater than 2, then (heuristic *Prop. 1*) the triples {a,b,c} to the equipotent powers (from 1 up to *p*) constitute scalene acute-angled and obtuse-angled (but not right-angled) triangles. This is not enough for the proof; we must jettison the *Counter-hypothesis*. With the help of an *Auxiliary Construction* created on the heuristic insight of *Prop. 1* (but logically independent of it and from the *Counter-hypothesis*), we applied FT (1640) in its two variants to the formulae of the *Auxiliary Drawing* with prime exponents. The outcome is, on each rung, four formulae divisible by the prime exponent. They are all equivalent without loss of generality, iff {a,b,c} are *not divisible* by the prime exponent. This applies to the *even prime* (2) also. Therefore, we proved that there is a self-contradictory trait in the *Counter-hypothesis:* for the exponent 2, the two variants of FT (1640) are equivalent *and* the four even formulae ensuing from their application are equivalent, iff {a,b,c} are all *odd*. Other alternative even formulae (obtained if 2|a or 2|b or 2|c) lead to contradiction by *reductio ad absurdum*: the formulae ensuing from FT (1640), albeit even, are not equivalent with one another, which is a loss to generality. This exhibits a self-contradictory trait in the *Counter-hypothesis*: it annihilates itself. Thus (*Prop. 4*) there are no pairwise relatively prime integers a>b>c>0 and no prime exponent *p* such that the negation of FLT 1637 is true. Owing to Fermat's previous (extant) proof to the Biquadratic Equation, this is sufficient to the full proof of FLT (cf. RIBENBOIM 1979, p. 36). Ours is a reconstruction of the heuristics of Fermat's own historical proof, completely independent of the modern proof of Andrew Wiles (1994).

The historical reconstruction of the heuristics of the proof is based on Fermat's Little Theorem (1640), Eratosthenes' «Sieve», and on a few propositions of the *Elementa* (especially VII.27, IX.29). Fermat called his method of proof the Infinite (or Indefinite) Descent. We would like to call it the Prime Number Induction; it is isomorphic with but different from, the mathematical induction.

The reconstructed proof to FLT (1637) is tantamount to proving that {a, b, c} are *odd*. Hence the sum of the *p*:th powers of (b) and (c) cannot be of the same parity as the *p*:th power of (a). Therefore, they are *not equal either*. The *Counter-hypothesis* is annihilated; it is a singularity not passing the test of parity. The parity of the formulae of the *Auxiliary Drawing* is tested by FT (1640) which makes them even. The parity relation (with its reflexiveness, symmetry and transitivity) is needed to compare the even formulae with one another, and to uncover contradictions hiding behind the parity.

Fermat's proof to FLT (1637) complements his extant proof to to the Biquadratic Equation. The two proofs have similar structures, too. Furthermore, the reconstruction to FLT (1637) gives rise to the «Prime Number Induction» which entails (by the κόσκινον, in the *Appendix*), that {a,b,c} are not divisible by the primes from 2 to *p*, both included. Hence the corresponding scalene plane triangles (implicitly defined in *Elem.* I.20) have *prime* sides. This leads to the 3-body problem in the sequel.

There are, of course, scalene triangles with pairwise relatively prime sides independent of the *Counter-hypothesis*. Yet the same method, with the employment of FT (1640), suffices to prove that the integer sides a>b>c>0 pairwise relatively prime of *all* scalene plane triangles {a,b,c} are prime. For this subset of plane triangles, FT (1640) is a «prime generator» in cooperation with Eratosthenes' «Sieve» (cf. our Appendix to Fermat's Marginal Note). This applies to any assigned prime exponent (from 2 to *p*), according to *Prop. 4*, but *p* is infinite for *equilateral* triangles alone, not for scalene plane triangles. Ultimately, it was Fermat's feat to create a universal method, FT (1640), to provide the κόσκινον with a driving force, without loss of generality. Together with the auxiliary construction **{H}**, it provides us with an *operational definition of the primes.*

Fermat's «Prime Number Induction» complies with Bernard Russell's view on induction, albeit a prime *p* and the previous integer (*p*-1) only, are immediately involved at each step (cf. our paper, Ch. «Historical parallels to Fermat's method»). The well-ordering of the natural numbers is not presupposed; the order of primes admits of varying intervals. An immediate successor occurs between the primes 2 and 3 only.

Goldbach's Conjecture is based on his letter to Euler 1742: «At least it seems that every number that is greater than 2 is the sum of three primes». This shows that Goldbach held 1 a prime and accepted three equal primes as well. Euler corrected the conjecture: «All positive even integers greater than or equal to 4 can be expressed as a sum of two primes» (the «strong» conjecture). «All odd numbers greater than or equal to 9 are the sum of three odd primes» is called the «weak» conjecture. Both improvements admit the occurrence of two equal primes, and the latter one even three equal primes in 9=3+3+3. This suggests that the primality and parity of the summands was important; their unequality was not.

Euler also invented and proved two theorems on the generating functions which enumerate the partitions of an integer *n* into parts restricted in various ways; his method presupposed the introduction of an auxiliary second parameter and was based on power series (see *Hardy& Wright*, Ch. XIX). The theorems can also be proved «graphically», by arguments independent of the theory of infinite series. Euler did not restrict his focus on partitions expressed as sums of three different primes, however.

We shall focus on the variant of Euler where equal primes are excluded: «Every integer $n > 17$ is the sum of exactly 3 distinct primes». The few cases for smaller *n* can easily be numbered. This variant implies that the smallest and only even prime 2, occurs in every sum of three primes that represent *even* numbers. As c=2 is the smallest side of all triangles with integer sides, the variant excludes all *plane triangles* with pairwise relatively prime sides [by *Elem. i. 20*, c=2 > (a-b)=1, and hence either (a) or (b) must also be even, and {a,b,c} are not pairwise relatively prime]. Cf. RIBENBOIM (1979), p. 70, for a statement equivalent to FLT (1637) about a right-angled triangle with the smallest side c=2.

However, plane triangles with odd prime sides obtained in *Prop. 4*, constitute tripartitions into unequal primes, that satisfy Goldbach's Conjecture in the variant selected (when $n > 17$). If we denote n=a+b+c, a>b>c>0 unequal primes, there are also other partitions into three dintinct primes such that either a=(b+c) or a>(b+c). They may represent triangles with integer sides a>b>c>0 on spherical, parabolic, elliptic, or hyperbolic surfaces. Thus they are, basically, analogues of the Euclidean scalene triangles on plane (implicitly defined by six inequalities at *Elem. i. 20*).

As to Goldbach's Conjecture, we can suppose without loss of generality, that the three integers a>b>c>0 making up n = a+b+c > 17 are *pairwise relatively prime* and, in particular, c=2 in the case n is even.

Now the particular feature of plane triangles, $a<b+c$ (*Elem.* I.20), must be replaced by $a>b+c$ or by $a=b+c$ (the latter for *even n* alone). In both cases, the *method of proof* employed in *Prop. 4* is applicable again. In Goldbach we focus on a positive problem of *all triples* with pairwise relatively prime integers, and we must modify the method accordingly. Fermat himself tells about his prolonged efforts to apply his method to positive assertions, in contradistinction from negative ones such as FLT (cf. *Fermat's Marginal Note*, Ch. *Fermat Verbatim*).

In the proof to FLT (1637) we introduced an auxiliary construction **{H}** as a touchstone for divisibility by the primes (from 2 to *p*). In the case of Goldbach's Conjecture, we introduce the auxiliary device, call it **{G}**, beginning with (a-b-c) and rising to (equipotent) powers of {a, b, c} up to the prime *p*. From the prime 2 onwards, the difference is positive; if $a=b+c$, then the square of (a) = the square of (b+c) > the square of (a) + the square of (b).

In their reprint with corrections (1988), Hardy and Wright update in an *Appendix* the Chapter «Unsolved problems concerning primes» including Goldbach's Conjecture in these problems. They sum it up: «none of these conjectures has been proved or disproved in the intervening 40 years» [since the first edition (1938)]; cf. n. 5.

New light is shed on Goldbach's Conjecture from the mathematical methods dealing with the *three-body problem*. In particular, Jeff Xia of Columbia University showed in his dissertation (1988) that already a system with four bodies (two massive bodies with a small satellite each) can be construed such that it will break up in finite time, one of the small satellites escaping infinitely far away from the rest (cf. D. G. Saari and Z. (J.) Xia 1995: «Off to Infinity in Finite Time», for a comprehensive review). That emphasizes the special status of triples. Xia's approach, however, brings along questions about the Newtonian celestial mechanics as well as about the triples of quarks in the Standard Theory. Should these physical theories be considered as ontology for Goldbach's triples of primes? Or else, should Goldbach's sums of three unequal primes be considered as models for the three-body problem solutions?

I submit that this is the most promising continuation to our historical reconstruction of FLT (1637).

It seems that Nature is prone to create triples of relative stability (even if not eternal) on all levels, from the quarks of the Standard Theory to the suns and their satellites in the starry heavens. On the other hand, by our reconstruction of the proof to FLT (1637), we have opened the gate to scalene triangles on plane with three distinct primes as their sides. Goldbach's Conjecture (in the variant where $n > 17$) will enhance this set of triples of distinct primes. For every number $n > 17$ admits of different partitions into three unequal parts, and if they are pairwise relatively prime, they can serve as models for the triples met in Nature. Geometries other than the Euclidean are then requested.

We believe that even the Pythagoreans, who created a synthesis of the naked-eye geocentric astronomy, theory of music, and arithmetic (cf. n. 6) would have accepted a synthesis based on triples of distinct primes. The idea, that a structure based on triples of primes describes Cosmos, strongly touches the mathematical mind today. But it is barely the first herald of a new worldview. Once the integers and their ratios touched the Pythagoreans. In the future, after the Bomb, perhaps Dolphins will play with pebbles on the beaches of the Aegean, maybe even near Cnidus?

A Selected Thematic Bibliography

(1) Previous works of the endeavour: Erkka MAULA and Eero KASANEN 1989: Chez Fermat A.D. 1637, *Philosophica* vol. 43, Ghent (Recent Issues in the Philosophy of Mathematics, ed. J-P. Van Bendegem); Erkka MAULA and Eero KASANEN 2004: From Non-Pythagorean Triples to Geeometry: A Hidden Lemma, *On the Edge of Fuzziness*, Acta Universitatis Lappeenrantaensis (LUT = Lappeenranta University of Technology, Finland), eds. Niskanen, V.A. and Kortelainen, J.; and Erkka MAULA, Jorma MATTILA and Eero KASANEN, *Fermat's Marginal Note and its Greek Historical Proof*, (70 pp. + 2 Auxiliary Constructions and 2 Drawings, unpublished paper).

(2) General orientation of FLT (1637) in modern mathematics before Andrew Wiles 1994: Paulo RIEBENBOIM 1979: *13 Lectures on Fermat's Last Theorem* (Springer, N.Y.-Heidelberg-Berlin) is a comprehensive and insightfull discussion of the problem in the mainstream mathematics; no historical reconstruction is attempted and Fermat's method is not analysed; a link from Wiles toward Fermat is built in Yves Hellegouarch 2002: *Invitation to the Mathematics of Fermat-Wiles* (2nd edition, Bodmin). It is an admirable work in many respects, and contains good observations of Fermat's own descriptions of, or at least hints at, his method of Infinite (or Indefinite) Descent. Unfortunately, no historical connection is established between the two giants. Alas, an analysis of the logical connections between the different branches of mathematics, which they represent, also remains to be made in the future; Fermat's Little Theorem (FT 1640) and its immediate consequences are lucidly analysed in HARDY, G. H. and WRIGHT, E. M. 1996: *An Introduction to the Theory of Numbers* (5th ed. Oxford); the connection of FT (1640) to FLT (1637) is not discussed, however. We also have reason for criticism concerning their interpretation of Fermat's extant proof to the Biquadratic Equation: owing to the modern technique of proof adopted, Fermat's obvious geometrical predilections are omitted or diluted.

(3) From history to heuristics to proof: Applied for thirty years in our interdisciplinary research, our strategy was put into writ in my paper, Proof,

History and Heuristics, *Historia Scientiarum, 29*, 1985, Tokyo. FLT (1637) is discussed in the context of Hilbert's *Problems*. For in 1900 Hilbert apparently considered the solution to FLT (1637) a paragon among all Diophantine equations, maybe among his twenty-three problems for the mathematics of the beginning century, too. See also my review of Alexandrov et al.: *Die Hilbertschen Probleme* in *Zentralblatt fuer Mathematik 568*, Berlin. For a mathematician turned historian, however, the urge to replace a meandering history by logical map, or the crossing paths of a garden of heuristics by a straight-forward evolution, is compelling, as in B. L. van der Waerden. Die Postulate und Konstruktionen in der fruegriechischen Geometrie, *Arch. Hist. Exact. Sci.*, 18, 1978. Cf. also my review in *Zentralblatt für Mathematik 432*, Berlin, which led to a long correspondence and many discussions in the Tagungen in Oberwolfach, and elsewhere.

(4) Is invention automatical? Cf. my review essay 1981: An end of invention, *Annals of Science* vol. 38, No. 1, London. Analysis-and-Synthesis is not a method of invention, far less a means of automatic theory-proving as Prof. Jaakko Hintikka and Dr. Unto Remes contend in their book, *The Method of Analysis. Its Geometrical Origin and Its General Significance* (*Boston Studies in the Philosophy of Science* vol. XXV, Dordrecht, 1974).

On specific conditions, however, it is not merely a heuristic means, an adjuvant to proof, but also a method of proof by *reductio ad absurdum*. This concerns the theoretical kind of analysis, and applies to Fermat. The Greek method does not make auxiliary drawings and constructions superfluous. They belong to the κατασκευή of Greek mathematics and have an independent role, which cannot be substituted for by logical reasoning. Fermat's method of Infinite (or Indefinite) Descent presupposes an auxiliary construction **{H}**. In turn it provides us with a new type of inductive proof, unknown to the canons of modern logic and indirect proof. Recall that Hamilton tried to solve the problem of interger triples for twenty years in vain, according to M. KLINE. *Mathematical Thought from Ancient to Modern Times*, New York, 1972. Our unpublished work in fact explains why Hamilton could not find the arithmetic for the integer triples.

(5) From Fermat to Newton: the 3-body problem: Hardy and Wright highlight the works of VINOGRADOV (1937), RIESEL and VAUGHAN (1983), and CHEN 1975: *Sci. Sinica 18* [see also *Sci. Sinica 16* (1973) and *21* (1978), and CHEN, J.R. and WANG, T.Z. 1989: «On the Goldbach Problem», *Acta Math. Sinica, 32*], summing up that «all the results mentioned in this paragraph have been found by the modern sieve methods». For these, cf. H. HALBERSTAM and H.-E. RICHERT, *Sieve Methods*, London, Academic Press, 1974. No reference was made to the interplay of the «Sieve Methods» and FT (1640), or to the connection of Goldbach's Conjecture with the then unsolved FLT (1637). That is the common wisdom in the net, these connections are

not acknowledged; cf. E. W. WEISSTEIN: Goldbach Conjecture, *Math. World* http://mathworld.wolfram.com, with an extensive bibliography. The new approach was heralded by D. G. SAARI and Z. (J.) XIA in May: Off to Infinity in Finite Time, *Notices of the AMS*, vol. 42, No. 5, 1995, pp. 538-546.

(6) Synthesis: theory of numbers and the heavens from Pythagoreans to Fermat to Newton: our reconstruction rises from the synthesis of ancient Greek mathematics, astronomy, theory of music and philosophy, cf. A. SZABÓ et E. MAULA: *Les débuts de l'astronomie, de la géographie et de la trigonométrie chez les grecs*, Paris, Vrin, C.N.R.S., 1986 and A. SZABÓ: *Anfänge der griechischen Mathematik*, Budapest, 1969. For the Pythagoreans in particular, cf. my paper: The Conquest of Time, *Diotima* 11, 1983, and our first common paper 1976: The Spider in the Sphere/ Eudoxus' Arachne, *Philosophia* V/VI, Academy of Athens.

(7) A general bibliography: in alphabetical order is given at the end of our paper: Fermat's Marginal Note.

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Η ΧΑΜΕΝΗ ΑΠΟΔΕΙΞΗ ΤΟΥ FERMAT ΚΑΙ ΟΙ ΑΡΧΑΙΟΙ ΕΛΛΗΝΕΣ ΜΑΘΗΜΑΤΙΚΟΙ: Η ΙΣΤΟΡΙΑ ΚΑΙ Η ΦΙΛΟΣΟΦΙΑ ΤΟΥΣ

Περίληψη

Στὸ κείμενο αὐτὸ παρουσιάζεται σὲ περίληψη μιὰ εἰκασία γιὰ τὴν πρωτότυπη ἀπόδειξη τοῦ Θεωρήματος τοῦ Fermat. Ἡ ἀπόδειξη βασίζεται στὴν μέθοδο τῆς Ἀναλύσεως-καὶ-Συνθέσεως, τῶν Ἑλληνικῶν Μαθηματικῶν, χρησιμοποιεῖ μία ἐπικουρικὴ κατασκευὴ **{Η}** ποὺ δὲν μπορεῖ νὰ ἐνταχθεῖ στὴν ἀπόδειξη καθαυτὴ καὶ μιὰ παραλλαγὴ τῆς Μαθηματικῆς Ἐπαγωγῆς, τὴν «Ἐπαγωγὴ τῶν Πρώτων Ἀριθμῶν», ὅπως ὁ συγγραφέας τὴν ἀποκαλεῖ.

Δούκας ΚΑΠΑΝΤΑΗΣ
(Ἀθῆναι)