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THE CONSTANTS OF NATURE

A STUDY IN THE EARLY HISTORY OF NATURAL LAW*

The concept of scientific progress is often characterized by saying that «what was previously considered a constant of nature, in later analysis turns out to be a variable» (cf. Ketonen [6], pp. 477-480). Although ambiguous, this maxim nevertheless points towards a number of relevant factors worthy of a closer examination in the history of science. First of all, it refers to the Greek idea of knowledge that is based on invariances discovered in nature and arranged into rational (or, at times, speculative) patterns. Secondly, it refers implicitly to the perpetual oscillation between permanent features in

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* This paper was read at the Research Center for Greek Philosophy of the Academy of Athens in May, 1974, at the invitation of Professor Johannes Theodoracopoulos. I hereby would like to pay homage to my teacher, Professor Oiva Ketonen, University of Helsinki, on the occasion of his sixtieth birthday. His book on the Great World Order [6] kindled my interest in the history and philosophy of the exact sciences. — References to the Bibliography are made in s q u a r e b r a c k e t s, references to formulae in r o u n d b r a c k e t s:

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[2] Dicks, D.R., *Early Greek Astronomy to Aristotle*, New York 1970
[3] Hare, R.M., *Plato and the Mathematicians*, in *New Essays on Plato and Aristotle*, ed. R. Bambrough, London 1965
[4] Heath, T.L., *Aristarchus of Samos*, Oxford 1913
[5] Heath, T.L., *A History of Greek Mathematics*, vol. 1-2, Oxford 1921
[6] Ketonen O., *Suuri Maailmanjärjestys*, Helsinki 1948
[7] Lasserre F., *Die Fragmente des Eudoxos von Knidos*, Berlin 1966
[8] Maula E., *Ancient Shadows and Hours*, in «Annales Universitatis Turkuensis», Ser. B, Tom. 126, Turku 1973.
[9] Maula E., *Studies in Eudoxus' Homocentric Spheres*, Helsinki (Comm. Hum. Litt. vol. 50 - Societas Scientiarum Fennica) 1974
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[11] Mittelstrass J., *Neuzeit und Aufklärung*, New York 1970
[12] Neugebauer O., *On the Hippopede of Eudoxus*, «Scripta Mathematica» 19 (1953)
[13] Neugebauer O., *The Exact Sciences in Antiquity*, Copenhagen 1957[2]
[14] Schiaparelli, G.V., *Le sfere omocentriche di Eudosso, di Callippo e di Aristotele*, «Publl. del R. Osservatorio di Brera in Milano» IX (1875).

the world-order and permanent features in the formulae that are supposed to express the natural laws, in the search for such invariances. And finally it brings into the concept of a scientific paradigm the important methodological aspect. For although the history of science is often done in the narrow sense of recording only the results of science, these to my mind should be contrasted with the contemporary world-view, scientific terminology, epistemological ideal, and analytical means.

While the quest of the cosmological and epistemological context of the superseded results justifies the philosophical investigations of a past science, terminological studies undertaken even in complete linguistic isolation do not seem to need any justification. But the modern historian of science still seems to be obliged to give reasons for investigations of past scientific methods, perhaps due to ancient doxographical traditions, which recorded the results but omitted the methods of science. Witness the history of the theory of homocentric spheres, to the beginnings of which this case-study is devoted.

But since the bearing force behind the ever renewed historical interpretations of past events is the accruing knowledge of the interdependence of the historical facts and the (improving) historical methods, I do not see any justification for the separation of the results from the methods in the discussion of other sciences either. True, this combines the interests of the historian of science with those of the philosopher of science, but I believe that both parties will gain from this combination.

1. The semination of the scientific law of nature.

Supposing there is a clear distinction between an invariable world-order and an invariable formula that is meant to describe this world-order, what would the first explicit formulation of a natural law look like?

We need not be concerned here with certain modern schools which tend to eliminate the whole distinction in favour of an entirely formal approach to all questions of order, for this was not the Greek way of thinking. Nor do we need to restrict ourselves to the extreme Parmenidean concept of strict immutability of the world-order even at the expense of explaining all change as illusory, although such a view (in a modified form not alien to some moderns) might be expressed by one invariable formula or diagram. Nor do we need to look for such doctrines alone which assume that the search for truth presupposes a structural isomorphism between the natural processes and the method of investigation (like the dialectics of nature and the dialectical method), or a functional isomorphism between the knower and the known (expressed e.g. in the maxim «like knows like» or any such phrase of the Socratic and Atomist schools).

Nay, we may start with the admission of changes in nature and nevertheless look for invariances in these changes, expressible in clear-cut formulae. Hence not even Zeno (nor Plato in the *Parmenides*) has fathered the concept of natural law, although he certainly has done much for the growth of the axiomatic method. What about the Atomists, then? Certainly the Vortex might have implied a distinction between the concepts of a period (T), angular speed ($\omega = 1/T$) and linear speed ($v = r\omega$), although this perhaps never can be documented. Likewise the concept of the resistance of the intermediary substance was capable of explaining a number of natural phenomena, including the Sun's longitudinal anomaly, the seasonal Mediterranean North-East winds and the brightness of the celestial objects (excluding the Moon). And we happen to know that the mathematician Hippocrates tried to axiomatize the Atomists' doctrines. But even if we take all these ingredients together, they do not amount to what might be called a natural law. At best, the conglomerate could be called a qualitative description; the descriptive element is simply too strong for the explicit formulation of a natural law, although it might have been judged adequate for the contemporary concept of world-order. For we can suggest a likely set of axioms which Hippocrates probably pondered: they have included references to the dualism of void and matter; a very great, or even infinite number of atoms; a finite number of kinds of atoms; a very great, or even infinitely great space; a very long, or even infinitely long duration of time; and so forth. We cannot be very far from the birth-place of the concept of natural law, however, for Democritus' auxiliary concepts include the celestial sphere, and there is the suggestion (VS 2, p. 141) that one of Democritus' works dealt with «projections of the armillary sphere on a plane».

It is the last point mentioned, the rational element and geometrical design exhibited in the world-order, that makes it futile, in my opinion, to try to extract any natural laws (in the modern sense of the term) from the Babylonian mathematical astronomy. True, there is an element of prediction and skilful arithmetical techniques, and the practical results are at times even better than the Greek ones. But, as Neugebauer has pointed out ([13], cf. p. 156), the Babylonian methods «nowhere point to an interpretation through a combination of circular motions or any other mechanical model»[1]. And the Mesopotamian mathematical tools, the zigzag and step functions, practical-

1. Neugebauer has also shown that Egyptian mathematics simply was not adequate for astronomical calculations of any complexity.

ly exclude such models. I am willing to admit, though, that this is a matter of taste. One can well consider such revisions of the concept of natural law as do not presuppose any underlying geometrical models. The idea of starting from «pure facts» might even appeal strongly to a bold spirit, although in that case the principles of the economy of thought and of the axiomatic system, and the concept of Protophysik, which have already shown their fruitfulness in the interpretation of Greek science, perhaps must be jettisoned. It is not unthinkable, however, that a discovery of one single clay-tablet may drastically change our views on Babylonian science in these respects.

For the reasons given above, I would suggest that the first «natural laws» are to be found in Greek science, and in the period after Democritus. This implies Pythagorean astronomy and mathematics as their conceptual environment — not Pythagoras' own teaching and not eve nthe Philolaic system, but rather the Platonic—Eudoxan research programme of «saving the phenomena». The postulates of uniform circular motion and constant angular velocities, where the diurnal rotation of the sphere of the fixed stars is the swiftest of all (for these assumptions see Mittelstrass, [10], [11]), adumbrate the axiomatic element. The celestial sphere accounts for the geometrical model. And Plato's insight into the role of mathematics (for which see Hare, [3]), together with the Academicians' and Eudoxus' contributions to, and refinements of, the contemporary (Pythagorean) mathematics, will provide the mathematical techniques and the formulae for the expression of a «natural law»[2]. I would like to add that although Plato on many grounds can be discussed separately from Eudoxus, the *Timaeus* fragmentary astronomy can, and should, be discussed with reference to Eudoxus' cosmology. Indeed it was

2. I would like to remind the reader here that I am using the terms «natural law», «laws of nature» and «law» in rather a modern sense, without claiming that Plato or Eudoxus ever used these t e r m s to denote the concept of natural law. On the other hand, I am inclined to think that, no matter which terms were used, e.g. *desmos* as in Plato, the concept of the laws of nature was properly understood by Plato and Eudoxus. Perhaps I may quote from a letter of Professor Johathan Hodge, University of Pittsburgh. «My only reservation is about your phraseology, and concerns the use of the word «law». It may well be that the Greeks believed, enunciated and used many propositions that would meet any reasonable criteria for the application of the term «law». However, this would have to be argued for explicitly, surely, given that they very rarely invoked the l e g a l m e t a p h o r in asserting and specifying orderliness in natural changes. I have not found the literature on «law», «natural law», «law of nature» etc. very helpful in this regard, since it does not really confront the key issue directly: Were those, like Philo and the Stoics, who first systematically explicated their convictions about nature's orderliness in terms of the legal, or better, the constitutional analogy, were they simply giving a new gloss on an old doctrine or did they transform it in the process of rephrasing it?»

my working hypothesis in [9], which has turned out to be quite fruitful, that Plato's frame of reference was Eudoxus' theory of the homocentric spheres. Now Eudoxus' theory was developed further mainly through generalization. Hence the scientific progress within this paradigm presumably could be outlined with reference to the maxim about the constants and variables in the explicit formulation of a natural law. On the other hand, if we start from the later formulations of Eudoxus' theory — from Torriano in the XVI Century, through Ptolemy, Hipparchus, Apollonius, Eratosthenes, Aristotle and Callippus — and return to Eudoxus again, we must ask whether all later variables are reduced step by step into constants. At least some of them do reduce. But alterations have been made also with respect to the axiomatic element, the models, and the computing techniques.

Nonetheless the degeneration of variables into constants is a useful measure, since it indicates the accuracy of observation and the length of observation series. In Ptolemy and Hipparchus the eccentricity of the solar orbit for instance remains a constant that was included in their laws of nature in the planetary theory. In fact the eccentricity is not a constant, but the period of its regular change was too long (T = 96000 years) for the ancient instruments. Again the obliquity of the ecliptic was considered a constant by the ancients[3]. Its regular change within certain limits ($21^0 39' \leq \varepsilon \leq 24^0 36'$) was not observed, because of the long period (T = 41000 years). And to take a third example, the motion of the solstitial and equinoctial points was not discovered before Hipparchus, although it can be observed in far shorter periods (50.26″ p.a. or 1^0 in about 72 years; yet Ptolemy still used a constant: 1^0 per century, *Syntaxis,* vii, 2). This means in fact that no earlier observation series could be consulted with respect to star-maps, although observations of the planetary periods (of the Babylonian type, where random errors in the long run balanced each other) may have suggested rounded-off values.

As we noted, the progress made within the paradigm of the homocentric spheres touched also features other than periods of the theory. Hence we must look for quite different constants in Eudoxus' system, too. Changes in computing techniques, for instance, may have contributed to the gradual abandoning of the predilection for integers and simple ratios in Pythagorean mathematics. Of this predilection there are clear indications also in Eudoxus. And changes in the model (the celestial sphere with 26 nested planetary spheres) and in the axiomatic element, made it possible for Eudoxus' followers

3. Or do we hear the first voices of doubt in certain explanations of the Milky Way as the Sun's «original route»? (Aëtius III 1, 2, cf. Heath [4], pp. 117-118). Professor Holger Thesleff informs me that if these ideas were Pythagorean, they probably derive from the mythological *akousma* tradition and not from the «mathematical» sect.

to abandon certain Eudoxan constants, for instance the equality of the astronomical seasons. Eudoxus' equalization meant jettisoning Euctemon's and Meton's discoveries of the inequality of the seasons (given in *Ars Eudoxi* as 90, 90, 92 and 93 days; the modern values to the nearest whole day are, 92, 89, 90 and 94 days) made some sixty years earlier. And this is just one example of Eudoxus' omissions of (direct) observational data, for which he was criticized even in ancient times. On the other hand, considering Eudoxus' achievements in mathematics (see e.g. [12]) and mathematical geography (see e.g. [8]), it is not warranted to simply record these omissions, but we must study to what extent they derive from his method. Likewise we must study how long observation series would have shown that they a r e omissions. With these broader aspects in mind I shall discuss Eudoxus' method and the considerable number of constant values ascribed to his system in tradition.

And lest the more general historical perspectives be forgotten, it is as well to remember the depth of Eudoxus' mathematical insight[4]. Dedekind's theory of the irrational numbers draws directly on Eudoxus (Eucl. v, *Def.* 5), and there is a direct line from Eudoxus to Weierstrass, too (cf. Heath [5] i., pp. 326-327). Moreover, it is well known that, through Archimedes, Eudoxus has begun anotner line of development, which leads to integral calculus. In these directions Eudoxus' results fertilize much later mathematical thought, because his methods, the theory of proportion and the method of exhaustion, were known, too. It is different with Eudoxus' astronomical results: Schiaparelli [14] had to substitute his own conjectures for Eudoxus' parameters, since he did not know Eudoxus' method.

2. On the background of Eudoxus' method.

I have outlined the reconstruction programme for Eudoxus' theory in my studies in the homocentric spheres [9]. It amounts to claiming that all previous attempts have failed because they have started from the scattered parameter values attached to the system. Instead, one should begin with the reconstruction of Eudoxus' method and computing techniques and see whether these parameter values can be obtained through their medium. For various reasons, which it is unnecessary to repeat here, I suggested that there indeed is one computing technique which is capable of explaining both the known unproblematic features of Eudoxus' system and, in addition to these, its known oddities and idiosyncrasies, e.g. the fictitious deviation postulated

4. For a discussion of Eudoxus' importance in the history of mathematical analysis see e.g. my paper *The Elements of Analysis*, Proceedings of the XIVth International Congress of the History of Science, 19-27.8.1974, Tokyo and Kyoto, Japan.

for the third solar sphere (for the discussion of this feature by other commentators see Lasserre [7], pp. 201-203). The computing technique in question is at our disposal as soon as we have means of solving a generalized quadratic equation (which in fact gives the periods of two combined spherical motions studied in terms of their plane projections), conceived of as a proportion:[5]

$$(1)\quad \frac{xy}{x \pm y} = \frac{T^{comb}n}{n}$$

(where T^{comb} is the period of the combined motion and n an integer or rational fraction)

I would now make the additional claim that the proportion (1) is the explicit formulation of the only «natural law» needed in Eudoxus' system.

Although (1) is superficially similar to Apollonius' pivotal formula (see Ketonen [6], pp. 130-143), which can be given in the form

$$(2)\quad \frac{T^{syn}\,T^{sid}}{T^{syn} + T^{sid}} = T_{Sun}$$

(where T^{syn}, T^{sid} are the synodic and sidereal periods of a planet and T_{Sun} the period of the Sun)

there is no other explicit connection except this formal resemblance between (1) and (2). We shall not discuss in this case-study whether there is a historical connection between them, discoverable in a more general survey. How is (1) constructed? Leaving aside the problems of the transmission of Babylonian scientific knowledge into Greece about one generation earlier than Neugebauer presumes (see [13], esp. pp. 150-151) — for in my opinion they will be satisfactorily answered by my results — I submit that the starting-point has been the so-called «normal forms» of Babylonian mathematics. In these «normal forms» (Neugebauer's term) two numbers should be found when (a) their product and (b) their sum or difference is given:

$$(3)\quad \begin{cases} xy = a \\ x \pm y = b \end{cases}$$

Transforming the two equations of (3) into two linear equations

$$(4)\quad \begin{cases} x \pm y = b \\ x \mp y = \sqrt{b^2 \mp 4a} \end{cases}$$

the solution follows (see [13], p. 41) as

$$(5)\quad \begin{cases} x = \frac{b}{2} + \sqrt{\left(\frac{b}{2}\right)^2 \mp a} \\ y = \pm\frac{b}{2} \mp \sqrt{\left(\frac{b}{2}\right)^2 \mp a} \end{cases}$$

From the two equations of (3) we obtain

$$(6)\quad \frac{xy}{x \pm y} = \frac{a}{b}$$

5. There is no obvious term for equation in Eudoxus, but «proportion» of course is a commonplace. Alternatively we could start from the concept of a (generalized) harmonic mean; cf. [7], pp. 175f.

which is, of course, an obvious step as soon as (3) is known. Moreover, (6) can be conceived of as a proportion.

But there is only one solution to (3). In order to gain more freedom of operation, it is quite natural to generalize (3) or (6) by multiplying both *a* and *b* by *n* (where *n* is an integer or a rational fraction). It is a natural step for Eudoxus especially, since it can be based on the *Elementa,* v. 15, which, according to all traditions, belongs to Eudoxus' contribution to Euclid. But making now $a/b = T^{comb}$ we obtain the proportion (1). That is to say, we have introduced an auxiliary parameter *n* into the proportion (6). What does (1) express, then? Before the methods of spherical trigonometry were introduced (say between Menelaos' day and Theodosius' time), either only qualitative results were obtained or else graphical methods were used in the studies of spherical motions. As Neugebauer says ([13] p. 161), one of these seems to have been based on the discovery that stereographic projection of the sphere maps circles into circles. Hipparchus, who had no spherical trigonometry at his disposal, may have solved spherical triangles by the method of stereographic projection. More modest problems of circular motions in sphere, however, may have been studied even much earlier by means of their projections to a plane. Witness the work of Autolycus of Pitane[6]. This is still only one generation after Eudoxus. But then we have the title of Democritus' work mentioned above, which suggests that problems similar to those of Autolycus were discussed even before Eudoxus. If indeed «armillary spheres» of any complexity were constructed[7], the discovery that stereographic projection maps circles into circles would have been almost inevitable. But we shall see that even far simpler graphical means, viz. the cross sections of the celestial sphere, will suffice in the analysis of Eudoxus' circular motions. I am inclined, therefore, to follow Dicks here and to drop too advanced and anachronistic instruments from Plato's table— leaving only a s p h a i r i o n at most (cf. *Ep.* ii, 312 d). So I assert that this was enough for Eudoxus, and that (1) gives the period of the combined motion of two other motions (which are combined) characterized by their angular speeds, provided we can «solve» the proportion (1).

But what does one imply by saying that two motions are c o m b i n e d and characterized by their angular velocities? It is perfectly in order for a classical scholar to maintain that «Eudoxus combined the planetary motions

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6. Written perhaps about 330-300 B.C.; cf. *The Books of Autolykos: On a Moving Sphere and On Risings and Settings,* ed. and tr. F. Bruin and A. Vondjidis, Am. Univ. in Beirut, Beirut 1971.

7. See Cornford's opinion [1], pp. 74, 135; Wilamowitz, *Platon,* II, p. 390; and Apelt's *Platons Dialoge Timaios und Kritias...* n. 89, p. 163; but consult also Dicks [2] p. 120 sq.

in the sphere», but he must also face the logical consequences of this statement. If any two planetary motions are combined and computed, this means, in Eudoxus' day, that their projections are drawn on a plane, and the computations are made with reference to their angular speeds. This is a very difficult research programme, unless sufficiently strong analytical and synthetic means can be provided. Let us take the combination of the motions of the first and second Eudoxan spheres of any planet as an example. Because the diurnal westward rotation of the fixed stars, represented by the first planetary spheres, is the swiftest motion of the system, and because the second spheres (according to the Eudoxan tradition) are credited with an eastward motion, their combination is of the following type:

$$(7) \quad \omega_{I}^{ind(W)} - \omega_{II}^{ind(E)} = \omega_{II}^{comb(W)}$$

I have used here the indices (ind, comb) to denote the individual and combined motions. Since the rotation of the sphere of the fixed stars is not caused by any other outer motion or agent in Eudoxus, the first sphere is credited with the individual motion alone. The directions of rotation are also indicated (W or E), and I reserve the negative sign for eastward motions. Subindices (I, II) point to the first and second sphere. Now, since $\omega = 1/T$ (cycle per period), we obtain from (7) the form :

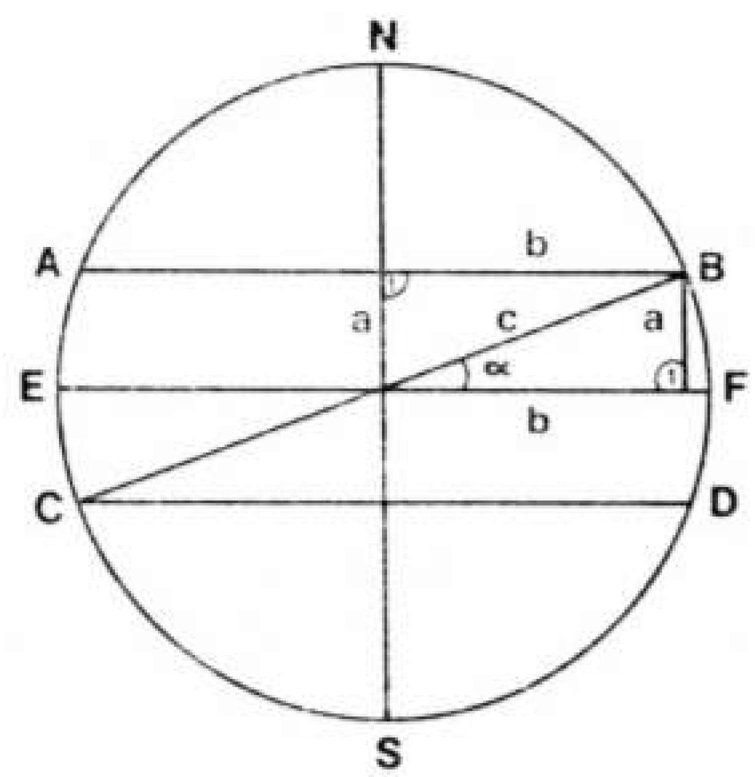
$$(8) \quad \frac{1}{T_{I}^{ind}} - \frac{1}{T_{II}^{ind}} = \frac{1}{T_{II}^{comb}}$$

where the directions of rotation have been omitted. But (8) is tantamount to part of the proportion (6), which can be generalized so as to obtain (1).

The result of this analysis of combined motions is one of the cornerstones in [9]. It amounts to the assertion that all second, third and fourth planetary spheres in Eudoxus, in so far as they are combined, must be credited with two motions (one individual, another combined) and hence also with two periods. But the extant Eudoxan tradition ascribes one motion and one period only to each planetary sphere. Hence either one must credit Eudoxus with too advanced mathematical methods, or drop the idea of combined motions, or else be prepared to account for the «extra» motions and periods. Now there are in the extant Eudoxan traditions certain hints, which it is unnecessary to rally here, at these «additional motions and periods» in Eudoxus' lunar and solar theories. Having shown my hand, however, I shall simply proceed and see whether the results will warrant my claim.

Before going into a detailed analysis of Eudoxus' method, it is wise to remember that if (1) indeed is the only «natural law» in Eudoxus, it must be capable of explaining a great number of things. Not only the known planetary periods but also others only vaguely referred to, like the «long» lunar and solar periods implied by the «slow» motions of the respective third spheres. And in addition to these the directions of rotation known from tradition (from Simplicius on), the axial inclinations of the third and fourth spheres, and so forth. These I call s u r f a c e p a r a m e t e r values. They in turn depend on what I call s t r u c t u r a l p a r a m e t e r values. The structural parameter values include the Eudoxan value for the obliquity of the ecliptic (which cannot have been based on direct g n o m o n observations, since the Sun was credited with a fictitious additional deviation from the ecliptic). That obviously restricts the choice of the axial inclinations mentioned. The traditional periods, too, may depend on one or several other periods (considered more fundamental for some reason) — be that the (unknown) Eudoxan luni-solar cycle, or Plato's «perfect year» (*Tim.* 39), should my working hypothesis about the frame of reference in the *Timaeus* hold good, or yet something else. And also the pattern followed in combining planetary motions belongs to the structural features essential in the reconstruction of Eudoxus' system.

Now my claim that (1) is the sole «natural law» in Eudoxus implies that whatever hierarchy of parameters we take, it is crowned by (1), on which all structural parameter values depend. Since these parameter hierarchies are discussed in more detail in [9], I give just one example of their interesting connections. It is a fact that the Eudoxan obliquity of the ecliptic can be computed starting from $T = 30$ days for the month, and $T = 360$ days for the year, on which all other traditional planetary periods in turn depend. What are these particular connections? First, $T = 360$ days is the calendaric year and $T = 30$ days is the length of the calendaric «full» month. The s i d e r e a l periods are «one year» for Venus and Mercury, two for Mars, 12 for Jupiter and 30 for Saturn. Second, all s y n o d i c periods known from the tradition, can be conceived of in the following way. $T^{syn}_{Venus} = 570$ days $= \dfrac{360 + 780}{2}$ days or the arithmetical mean of the calendaric year and the true Martian synodic period. $T^{syn}_{Mercury} = 110$ days which, since the Greeks rounded off by simply cancelling the fractions (cf. [13], p. 68), can be made $110\frac{10}{13}$ days $= 2 \cdot \dfrac{2 \cdot 30 \cdot 360}{30 + 360}$ days or double the harmonic mean of the calendaric month and year. $T_{Mars} = 260$ days $= 1/3 \cdot 780$ days $= 4/3 \cdot \dfrac{30 + 360}{2}$ days or four thirds of the arithmetical mean of the calendaric month and year. And $T^{syn}_{Jupiter} = T^{syn}_{Saturn} = 390$ days $= 2 \cdot \dfrac{30 + 360}{2}$ or double the arithmetical mean of the calendaric month and year. The third point reinforces the second one. On an additive interpretation of Plato's «great h a r m o n i a» we obtain, as duly pointed out in [9], a right-angled triangle which determines an excellent value for the Eudoxan obliquity of the ecliptic. Its hypotenuse is equal to $39 = 3 + 9 + 27$ or the sum of Plato's «triple intervals», and its shorter sides equal $15 = 1 + 2 + 4 + 8$ or the sum of Plato's «double intervals», and 36. The triangle in question appears in the cross-section of the celestial sphere as follows.



(where $c = 39$, $a = 15$, $b = 36$, and $\alpha = 22°37'$)

Fig. 1. Cross section of the celestial sphere. AB is a diameter of the summer tropic, CD a diameter of the winter tropic, EF a diameter of the equator, CB a diameter of the ecliptic, N the North Pole, S the South Pole, and α the obliquity of the ecliptic.

In determining α the relative lengths of the sides of the triangle are needed. Hence we can equally well multiply all sides by ten and obtain ($a = 150$, $b = 360$, and $c = 390$). The full importance of this triangle will be seen later, but suffice it to say that $b = 360$ and $c = 390$ can be conceived of as geometrical representations of the Eudoxan periods for the Sun, Jupiter and Saturn, while $(c - b) = 30$ represents the lunar period of thirty days. But the same triangle, the preconceived geometrical scheme which made it necessary to postulate the fictitious solar motion, can be obtained also through the application of the rule for generating Pythagorean triples (see [13], p. 39).

$$(9) \quad \begin{cases} c = p^2 + q^2 \\ a = p^2 - q^2 \\ b = 2pq \end{cases}$$

(where p and q are arbitrary integers which are relatively prime and not simultaneously odd, and $p > q$)

Several ancient commentators have attested that there are Pythagorean triples in Plato, who knew either (9) or some more particular rule. If we take Plato's basic «triple» and «double» intervals (3 and 2), and make ($p = 3$, $q = 2$), we obtain ($c = 13$, $a = 5$, $b = 12$) from (9). It may be noted that this is the simplest case where ($q \neq 1$). Multiplying the sides (a, b, c) by thirty we obtain the same triangle (c = 390, b = 360, a = 150) again. Hence there is in fact both a «Pythagorean» and a «linear» or additive genesis for it. This means, among other things, that at least some Eudoxan periods can be interpreted in terms of Pythagorean triples (and hence in terms of p, q), and also in the geometrical terms of line segments (a, b, c), and the same holds for at least some acute angles. In fact, we shall see that a l l Eudoxan periods can be so interpreted. This amounts to a synthesis of the Pythagorean and Hippocratic mathematics.

Perhaps these remarks will suffice for a discussion of the background of Eudoxus' method.

3. Eudoxus' method of analysis and synthesis.

I shall now discuss the mathematical character of the computing techniques needed in solving (1), taking Eudoxus' lunar theory as my starting-point. This might well correspond to the actual order of ancient studies, as suggested by Plato's educational programme for astronomy (cf. *Epin.* 990b, 990c-d, *Rep.* vii, *Tim.* 39c-d). Because (7) holds for the Moon, too, the combined motion of the second and third lunar spheres comes from one of the following formulae:

$$(10)\quad \omega_{II}^{comb\,(W)} + \omega_{III}^{ind\,(W)} = \omega_{III}^{comb\,(W)}$$

$$(11)\quad \omega_{II}^{comb\,(W)} - \omega_{III}^{ind\,(E)} = \omega_{III}^{comb\,(W)} \qquad (\text{where } T_{III}^{ind} > T_{II}^{comb})$$

$$(12)\quad \omega_{II}^{comb\,(W)} - \omega_{III}^{ind\,(E)} = -\omega_{III}^{comb\,(E)} \qquad (\text{where } T_{III}^{ind} < T_{II}^{comb})$$

If tradition (from Simplicius on) is correct about the w e s t w a r d rotation ascribed to the third lunar sphere, however, only (10) or (11) will apply. The lunar period T = 30 days, which we have discovered, may belong either to the second or to the third lunar sphere. Let us discuss first the case T_{III}^{comb} = 30 days. If so, tradition implies that T_{III}^{ind} is «long», since ω_{III}^{ind} must be «slow». Whether this will be so, can be seen by solving (1) with respect to T_{III}^{comb} = 30 days. We obtain the proportion

$$(13)\quad \frac{xy}{x \pm y} = \frac{30n}{n}$$

According to (5), the solution of the «normal forms» corresponding to (13) is

$$(14)\quad \begin{cases} x = \dfrac{n + \sqrt{n^2 \mp 120\,n}}{2} \\[2ex] y = \dfrac{\pm n \mp \sqrt{n^2 \mp 120\,n}}{2} \end{cases}$$

Let us discuss the negative sign case of (13) first. (i) If we are to obtain rational solutions, $(n^2 + 120\,n)$ must be a square. Let it be a^2. From this we obtain the equation $n^2 + 120\,n - a^2 = 0$. Hence $n = -60 \pm \sqrt{60^2 + a^2}$. Here again $(60^2 + a^2)$ must be a square; let it be denoted by c^2. Now the auxiliary variables a and c are obtained from the following Pythagorean triangle: $c = p^2 + q^2$, $a = p^2 - q^2$, and $b = 2pq = 60$ (see (9)).

Since $2pq = 60$, we must consider the following cases:

pq	p	q	α ↔	p/q
30	6	5	79° 37′	6/5
30	10	3	33° 24′	10/3
30	15	2	15° 11′	15/2
30	30	1	3° 49′	30/1

(where the acute angle α is given in degrees and minutes for the sake of convenience only, being determined by p and q in the following way: $\tan\alpha = 2pq/(p^2 - q^2)$. A paraphrase can always be given in terms of $q/p = \tan(\frac{\alpha}{2})$)

The lowermost case ($p = 30$, $q = 1$) implies indeed that particular form of (9) which Proclus (*On Eucl.* I, p. 487, 7-21; cf. Heath [5], i, p. 81) ascribed to Plato. If we choose it, we can draw the corresponding Pythagorean triangle diagrammatically as follows.

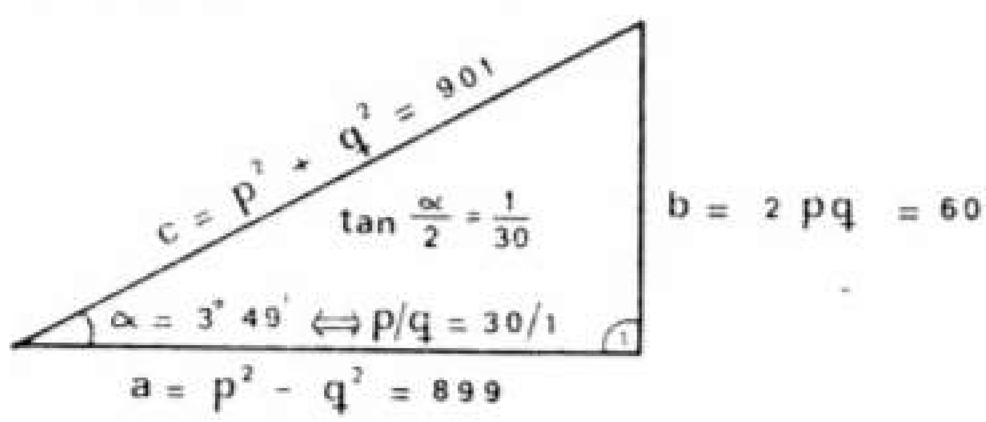


Fig. 2. A Pythagorean triangle for the tentative discussion of the third lunar (combined) motion in Eudoxus' system

The construction of this Pythagorean triangle is the culmination of the analysis of the lunar motions. The auxiliary parameter n has given rise to an auxiliary drawing of a synoptical character. We can now «turn backwards» and calculate first n (which could easily be illustrated by a diagram like Fig. 1). It is $n_1 = -60 + c = 841 = 29^2$ (while $n_2 = -961$, being a negative integer, may be omitted; cf. Heath [5], ii, p. 464). Hence from (14) $x = T^{ind}_{III} = 870$ days (which could well be the «long» lunar period), and $y = T^{comb}_{II} = 29$ days (which is the «hollow» calendaric month). And $T^{comb}_{III} = 30$ days (or the «full» calendaric month). In the last step we form the ratio $x/y = p/q = 30/1$. This is the end of the synthetic part. Now even the last vestiges of the auxiliary parameter n, introduced in the first step of the analytical part, have disappeared[8]. The geometrical by-product of the procedure is the acute angle $\alpha = 3^0\,49'$. It may be characterized in terms of q/p; $q/p = 1/30 = \tan(\frac{\alpha}{2})$. Presumably α is the axial inclination of the third lunar

8. Hence it is literally true that Eudoxus' system could not have been reconstructed from his results, but his method must be known, too.

sphere to the second one, which is equal to the Moon's maximum deviation from the (Eudoxan) ecliptic. For although ($22^0 37' + 3^0 49'$) is too small in comparison with the correct value in Eudoxus' day[9] ($23^0 44' + 5^0 08'$), the negative maximum deviation ($22^0 37' - 3^0 49'$) is a bit too great (as compared with the real value $23^0 44' - 5^0 08'$), and since the Eudoxan obliquity of the ecliptic was not defined in terms of the Sun's third motion, all planetary deviations from it necessarily have been compromises. If we choose any other combination of p and q, neither the axial inclinations nor the periods obtained will do.

The solution of the positive sign case of (13) follows the same pattern. (ii) If we are to obtain rational solutions, ($n^2 - 120\,n$) must be a square. Let it be a^2. From this we obtain the equation $n^2 - 120\,n - a^2 = 0$. Hence $n = = +60 \pm \sqrt{60^2 + a^2}$. Here again ($60^2 + a^2$) must be a square; let it be denoted by c^2. The auxiliary variables a and c are obtained from the same Pythagorean triangle as above (Fig. 2). Turning again backwards from this culminating point we compute n. It is $n_1 = 60 + 901 = 961 = 31^2$ (while $n_2 = - 841$ may be omitted). Hence from (14) $x = T_{III}^{ind} = 930$ days (which could well be the «long» lunar period), and $y = T_{II}^{comb} = 31$ days which, however, was not a length of a month in Eudoxus' time. Yet no other combination of p and q will give a better solution, either.

The point is, however, that the solutions (i) and (ii) of the «normal forms» corresponding to (13) can be generalized so as to apply to (1), and furthermore, in two ways. One generalization is in terms of p and q, and it might be called the Pythagorean Solution. The other generalization is in terms of the sides (a, b, c) of the right-angled triangle which is the culmination of the analysis, and it might be called the Linear Solution. I give first the two generalizations for a positive sign case of (1), and next for the negative sign case of (1)[10].

$$(15)\begin{cases} n = x + y = c + b \rightarrow (p+q)^2 \\ x = \dfrac{a+b+c}{2} \rightarrow p(p+q) \\ y = \dfrac{b+c-a}{2} \rightarrow q(p+q) \\ x/y = \dfrac{a+c}{b} = p/q \end{cases}$$

Solution of the «normal forms» $\begin{cases} xy = T^{comb}\,n \\ x + y = n \end{cases}$ constituting the proportion $\dfrac{xy}{x+y} = \dfrac{T^{comb}\,n}{n}$

Solution of the proportion

9. See Dicks [2], n. 240.

10. It may be noted that the solutions of x and y in terms of p and q are reminiscent of the well-known ancient Greek problem called «the application of the area».

$$(16)\begin{cases} n = x - y = c - b \rightarrow (p-q)^2 \\ x = \dfrac{a-b+c}{2} \rightarrow p(p-q) \\ y = \dfrac{a+b-c}{2} \rightarrow q(p-q) \\ x/y = \dfrac{b}{c-a} = p/q \end{cases}$$

Solution of the «normal forms» $\begin{cases} xy = T^{comb}\, n \\ x - y = n \end{cases}$ constituting the proportion $\dfrac{xy}{x-y} = \dfrac{T^{comb}n}{n}$

Solution of the proportion

Hence the final solutions are invariable with respect to the ratio p /q, which is a special ratio for each planet.

In both cases the two acute angles can be characterized also by means of their trigonometrical functions, i.e. as ratios of the gnomon to its shadow (see [8]).

$$(17)\quad \tan\alpha = \frac{2pq}{p^2 - q^2} = b/a;\ \text{and}\ \tan(90^0 - \alpha) = \frac{p^2 - q^2}{2pq} = a/b$$

but this is unnecessary, because the paraphrase in terms of q /p is enough. If $T^{comb} = p$ in (1) and $q = 1$, the Pythagorean Solution and the Linear Solution are equal (as in the case of the Moon). If not, they differ, but the triangle needed in the Linear Solution is always reducible to the triangle pertaining to the Pythagorean Solution (of the acute angles)[11] by simply multiplying all sides by a coefficient. Indeed, this was seen earlier when we discussed the triangles based on Plato's «great harmonia». Since the Moon is the simplest case from the methodological point of view (in other planets the two solutions do not coincide), it is reasonable to assume, as we did, that it has been the first target in Eudoxus' studies.

Considering the computing technique sufficient for solving (1), it seems warranted to divide the whole procedure into two parts. The first part consists in finding out the two triangles starting from the fact that double the period of a combined motion (T^{comb}) equals the side b of the triangle pertaining to the Linear Solution. This part might be called the Method of Analysis. Its culminating point is the construction of the two triangles. When these have been found out, the rest can be computed as shown in (15) and (16). The second part of the procedure might be called the Method of Synthesis. It ends with the forming of the ratio x /y = p /q, which we call the solution of the proportion. This use of language in my opinion corresponds to Hare's results in [3] and offers an interesting opening for future discus-

11. Eudoxus might have had access to tables listing or characterizing the angles in terms of q /p, but also direct observation will suffice, since $q/p = \tan(\alpha/2)$, when $\tan\alpha = 2pq/(p^2 - q^2)$, according to Eucl. vi. 3.

sions of the influence of Eudoxus' (and Plato's) mathematical methods on the contemporary philosophical methods of analysis and synthesis. As an example one might mention Plato's methods of the One and the Great-and-Small. I think that the above outlined Eudoxan methods, together with the Pythagorean method of approximating the surds, will throw light on them. After having read carefully Proclus and Pappus, I would also say that their accounts of the methods of analysis and synthesis are compatible with my reconstruction of Eudoxus, which might stimulate the current discussion. In this paper, however, I shall continue with an application of (15) and (16) to Eudoxus' planetary theories[12].

4. The Moon.

Starting from the traditional Eudoxan synodic periods we have discovered a lunar period of 30 days. Above, one alternative was discussed, viz. $T_{III}^{comb} = 30$ days. It remains to investigate the other alternatives, in which either T_{III}^{ind}, T_{II}^{comb}, or T_{II}^{ind} is eqLal to 30 days. I refer to the previous alternatives by (i) and (ii).

(iii) What about making $T_{II}^{comb} = 30$ days in (1)? If nothing more is known, the positive sign case is reduced to (i) and the negative sign case to (ii). It is easy to see, however, that the values of T_{II}^{comb} and T_{III}^{comb} can be interchanged, if the sign is changed, too. Consequently we must discuss also the proportion

$$(18)\quad \frac{870\ .\ 30}{870 + 30} = \frac{29\ n}{n},$$

which gives an even better value for the axial inclination: $p/q = 29/1$ or $\alpha = 3^0\,57'$ (while $\frac{930\ .\ 30}{930 - 30} = 31$ days gives a worse value $\alpha = 3^0\,42'$). I think that this is the correct solution.

(iv) There is one more possible solution which employs the integer lengths of the calendaric months, viz. $\frac{30\ .\ 29}{30 - 29} = 870$ days, but this implies an altogether implausible axial inclination: $p/q = 30/29$ or $\alpha = 1^0\,57'$. (And the positive sign case does not yield an integer solution).

(v) Next we may try to apply the «normal form» technique and experiment with the period $T = 30$ days, taking instead of the other lunar period of 29 days some other length of a month, e.g. $T = 27\frac{1}{3}$, $29\frac{1}{2}$ or $29\frac{1}{2} \pm$

12. In the Appendix my former student and present collegue, Mr. Eero Kasanen discusses my reconstruction of Eudoxus' method from an algebraic point of view.

$\pm \frac{1}{33}$ days (which were used in ancient times, but, as far as we know, not before Callippus; for more detail see [9], Table 2 and notes). But no likely results are obtained: n will not remain an integer, and the axial inclinations computed remain unlikely. (The relatively best result ensues from $T = 29\frac{1}{2}$ days or the mean synodic month; $n = 59^2/2$ and $p/q = 60/1$ or $\alpha = 1^0 55'$).

(vi) If $T = 30$ days does not belong to T_{III}^{comb}, T_{III}^{ind}, or T_{II}^{comb}, but to T_{II}^{ind} in (8), it can be shown that there will be no «long» lunar period at all for the third lunar sphere (all lunar motions being combined). For these reasons I conclude that (18) gives the correct solution, viz.

$$(19)\begin{cases} \text{The directions of rotation as in (10); } T_{II}^{ind} = 1\frac{1}{29} \text{ days — from (8);} \\ p/q = 29/1;\ n = 900;\ x = 870 \text{ days} = T_{III}^{ind}; \\ y = 30 \text{ days} = T_{II}^{comb};\ T_{III}^{comb} = 29 \text{ days; and } \alpha = 3^0 57' \end{cases}$$

It may be noted that the periods of 30 and 29 days need not be justified by calendaric reasons only. For since the Moon's motion in fact is not uniform (contrary to the Platonic-Eudoxan postulate), the length of the synodic month (i.e. the period between two consecutive new-moons) is not always equal to $29\frac{1}{2}$ days either. In fact the difference of two synodic months may be about thirteen hours. Hence the periods $T = 30$ days and $T = 29$ days may represent the upper and lower limits of the synodic month, accurate enough for practical (e.g. calendaric) purposes. Therefore (19) is compatible also with *Tim.* 39c, where Plato indicates the synodic month.

The two triangles corresponding to (19) are equal and can be represented diagrammatically in the following way.

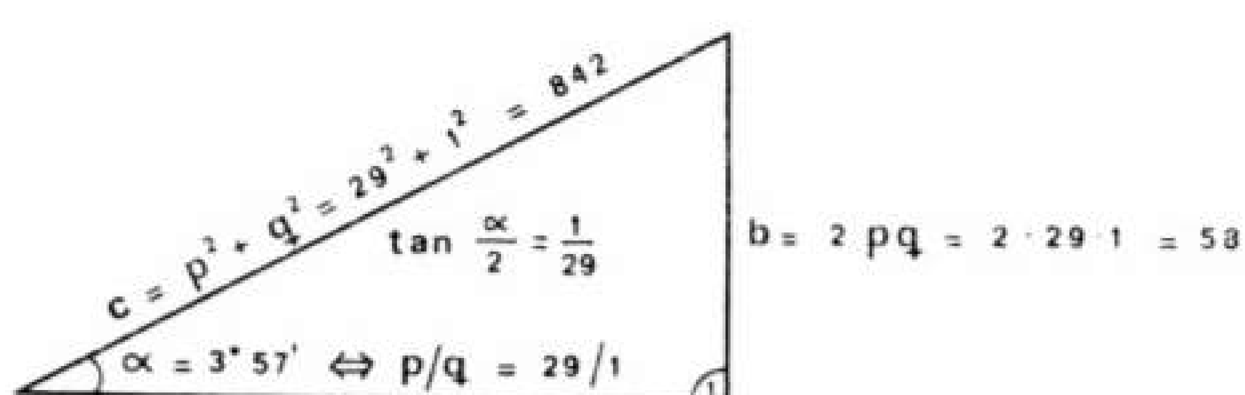


Fig. 3. The lunar triangle for the combination of the second and third lunar motions.

It can easily be checked that (15) holds good.

5. The Sun.

We have discovered a solar period of 360 days. Since in the Eudoxan tradition (from Simplicius on) the third solar motion is credited with an e a s t w a r d motion, the combination of the second and third solar motions cannot be of the type (10) if the tradition is correct. Hence the solutions do not come from the proportion

$$(20)\quad \frac{xy}{x+y} = \frac{360\ n}{n}$$

which implies westward motions only. But the combination cannot be of the type (12) either, because in that case T_{III}^{comb} would become «long», in contradistinction to Eudoxus' lunar theory, and an implausible axial inclination would be implied ($a = 89^0 22'$). Starting from the combination of the type (11), we obtain the solution from the proportion

$$(21)\quad \frac{xy}{x-y} = \frac{364\ n}{n}$$

where the period $T_{III}^{comb} = 364$ days (which has been found by solving (20))[13] might be called the «seasonal year». For it will be remembered that Eudoxus omitted Euctemon's and Meton's discoveries and e q u a l i z e d the astronomical seasons.

The two triangles pertaining to the Linear and Pythagorean Solutions are different from each other in the case of the Sun. They can be represented diagrammatically as follows.

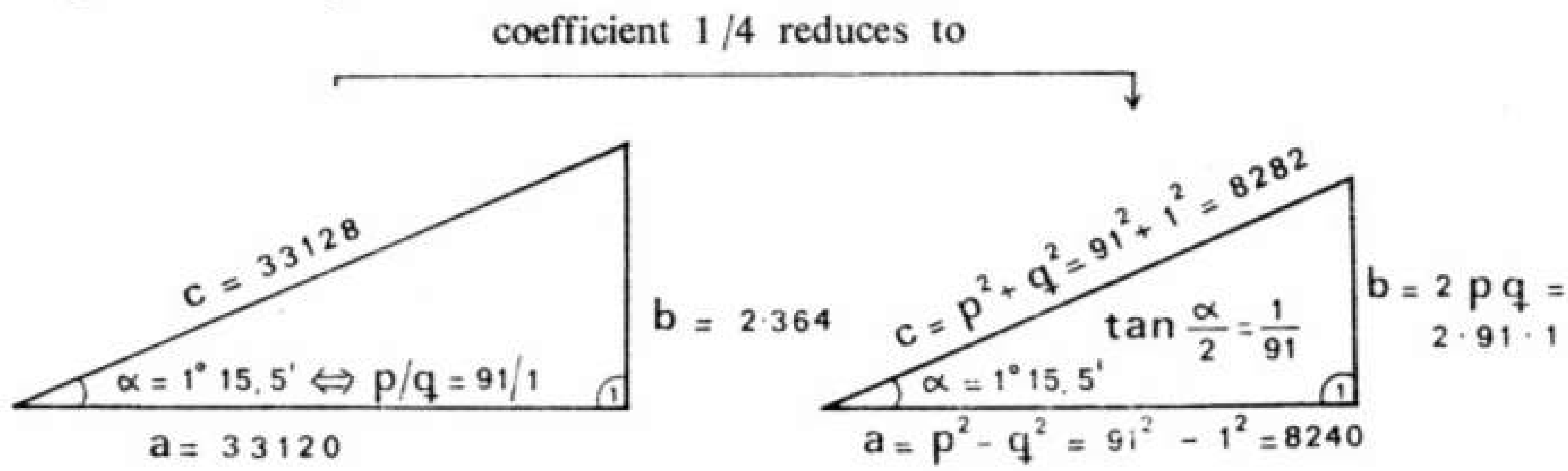


Fig. 4. The two triangles for the third solar motion; the Linear Solutions come from the left, the Pythagorean Solutions from the right

The solution is: $p/q = 91/1$, $n = 32400$, $x = T_{III}^{ind} = 32760$ days or the «long» solar period, $y = T_{II}^{comb} = 360$ days or the calendaric year, and $\alpha = 1^0 15.5'$. It may be noted that we now have reached one of the most im-

13. Censorinus associates a year of 364 1/2 days with Philolaus *(De Die Nat.)*. Being based on a progression of 3, just as Plato's triple intervals, this may tell about the contemporary discussions; 364 is the nearest integer if fractions are simply cancelled. — The solution cf. (20) with $p/q = 90/1$ gives $\alpha = 1^0 16'$.

portant results required. Eudoxus' m e t h o d implies the obtained fictitious deviation postulated for the third solar sphere. When it is added to the Eudoxan obliquity of the ecliptic, quite an accurate positive deviation in latitude is obtained ($22^0\ 37' + 1^0\ 15.5'$ as compared with the real value $\varepsilon = 23^0 44'$). And the negative (fictitious) deviation is still within the limits of Eudoxus' observational accuracy[14] (see [2], p. 155).

Furthermore, the seemingly superfluous axial inclinations postulated for the third planetary spheres now also make sense. For starting from (8) we obtain for the Sun by the same method as in the previous cases discussed, an axial inclination. But $T_{II}^{comb} = 360$ days for Venus and Mercury also (in a geocentric system)[15]. Hence the axial inclinations of their third spheres, the poles of which are on the ecliptical plane, can be understood to be the same as that of the Sun. This strange feature from the oldest tradition (Arist. *Met.* Λ 8, 1073b 30—32), too, is implied by Eudoxus' m e t h o d. For the other planets, these axial inclinations will be different in each case, and different from those of the Moon and the Sun. As for the periods obtained, no better value for the year will do. We cannot make, e.g., $T_{III}^{comb} = 365$ or $365\frac{1}{4}$ days (just as this type of operation was impossible in the case of the Moon). But it is as well to remember that Eudoxus complemented his astronomical work with calendaric studies[16]. Hence the «cosmological constants» obtained do not seem implausible.

6. The fourth motions of planets proper.

Since a h i p p o p e d e was created for each planet proper by the third and fourth planetary spheres, rotating in opposite directions in the synodic period (except for Mars which is credited with a period exactlyone third of the true synodic period; for an explanation of this see [9], pp. 82-83), their motions cannot be combined in the sense of affecting each other's periods. However, the motions of the fourth and second planetary spheres can be so combined (see [9], p. 112-113). What is more, due to the Eudoxan arrangement where

14. Eudoxus' observational accuracy can be estimated, thanks to Hipparchus' commentary *(Comm. in Arat.)* As regards the tropics, equator, arctic and antarctic circle, the error ranges from 1^0 - 3^0 in the majority of the cases. See Dicks [2], p. 155-156, but also [8]. It may be noted that x = 32760 days = 91 years of 360 days each, implies a fictitious solar motion (not to be confused with precession) of about 4^0 a year, and indeed there are indications of such a motion in Eudoxus. The period of 91 years meets our expectations concerning the length of Eudoxus' observation series : they are far shorter.

15. Cf. Dicks [2], n. 345.

16. See Dicks [2], pp. 188-189.

the axes of the third spheres of planets proper are situated on the ecliptic, the inclinations between the third and fourth spheres can be computed. The attached Fig. 5 will illustrate the arrangement.

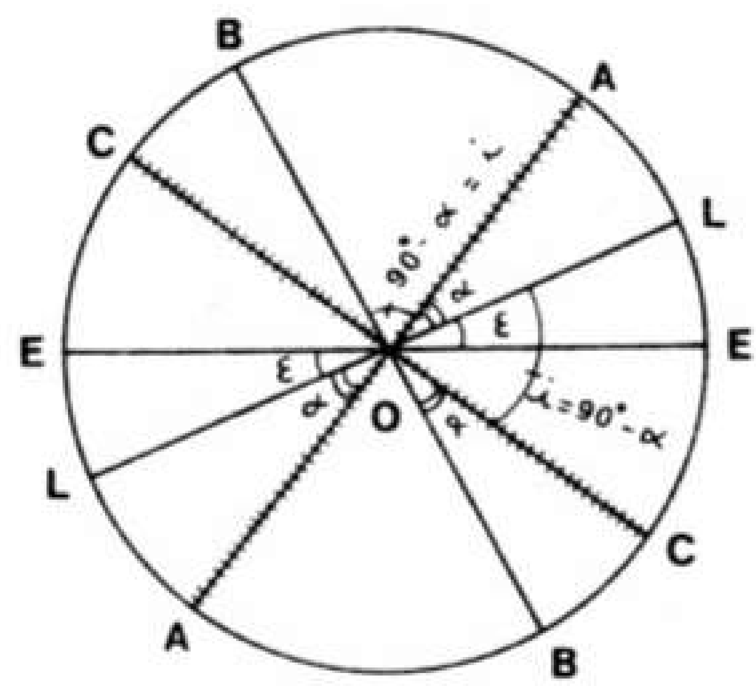


Fig. 5. A cross section of the celestial sphere. EE = the equator, LL = ecliptic = the axis of a planet's third sphere, BB = the axis of the ecliptic = the equator of a planet's third sphere, AA = the equator of a planet's fourth sphere (inclination to the ecliptic appears as α), and CC = the axis of the same planet's fourth sphere. Hence the inclinations *i* of the axis and equator of the fourth sphere to the axis and equator of the third sphere appear as the complement of α.

The computation of the periods of the combined motion of the planets' second and fourth spheres is made easier by the fact that both the sidereal and the synodic periods are known. Otherwise the computations follow the same pattern as in the cases of the Moon and the Sun. If the tradition (from Simplicius on) about the w e s t w a r d rotation of the fourth spheres is correct, the combined motions of the second and fourth spheres must be of the following types (and the lengths of the synodic and sidereal periods will determine which one applies for a given planet):

$$(22) \quad \omega_{II}^{comb\,(W)} + \omega_{IV}^{ind\,(W)} = \omega_{IV}^{comb\,(W)}$$

$$(23) \quad \omega_{II}^{comb\,(W)} - \omega_{IV}^{ind\,(E)} = \omega_{IV}^{comb\,(W)} \quad (\text{where } T_{IV}^{ind} > T_{II}^{comb})$$

Because the combined period of the fourth spheres in principle is observable, it is clear that it is equal to the planet's synodic period. Likewise the sidereal period is equal to the combined period of the second sphere. Hence the proportions, the corresponding two triangles, the solutions, and the inclinations of the fourth spheres to the third spheres are obtained as follows.

V e n u s The proportion is

$$(24) \quad \frac{xy}{x - y} = \frac{570\,n}{n}$$

The two triangles pertaining to the solutions are as follows.

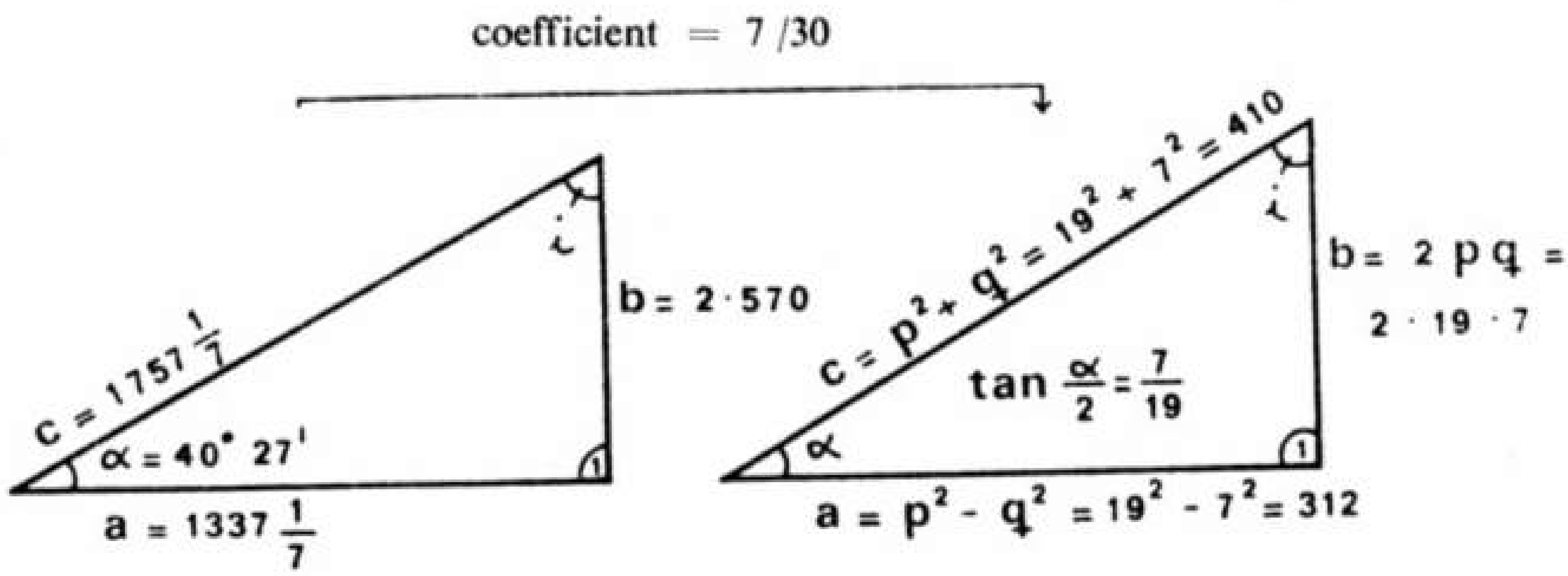


Fig. 6. The two triangles for the fourth motion of Venus

The solution is: $p/q = 19/7$, $n = 617\frac{1}{7}$, $x = 977\frac{1}{7}$, $y = 360$ days, $\alpha = 40^0\,27'$. Hence $i = 49^0\,33'$ (Schiaparelli in [14] gave $i = 46^0$).

Mercury The proportion is (for a westward rotation)

$$(25)\qquad \frac{xy}{x+y} = \frac{110\frac{10}{13}\,n}{n}$$

The two triangles pertaining to the solution are as follows.

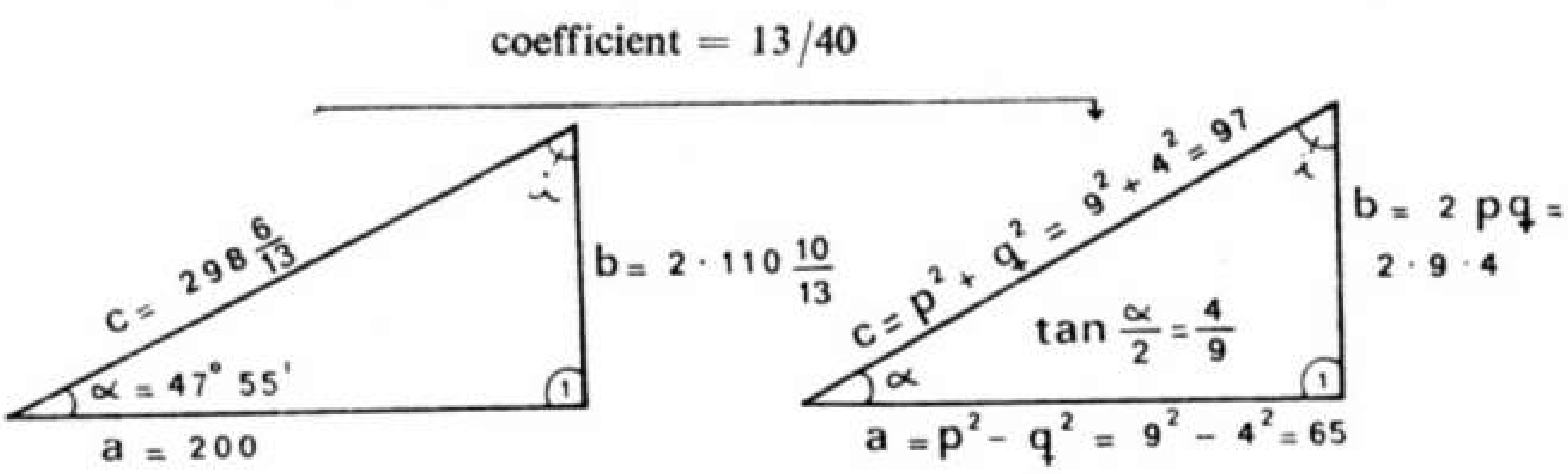


Fig. 7. The two triangles for the fourth westward motion of Mercury

The solution is: $p/q = 9/4$, $n = 520$, $x = 360$ days, $y = 160$ days, and $\alpha = 47^0\,55'$. Hence $i = 42^0\,05'$ (Schiaparelli gave $i = 23^0$).

It will be seen that, in contradistinction to Venus, the inclination i differs considerably from Schiaparelli's (conjectural) value. Since he had started from the assumption that i for Venus and Mercury represents these planets' maximum elongation, we may consider an alternative in the case of the direction of rotation of the fourth sphere of Mercury. If Simplicius is not right in crediting the fourth sphere of Mercury with a westward motion, but was led to do so because of the analogy of other planets, the combination of the motions of the second and fourth planetary spheres of Mercury may have been of the following type (cf. (12))[17]:

(26) $\omega_{II}^{comb\,(W)} - \omega_{IV}^{ind\,(E)} = -\omega_{IV}^{comb\,(E)}$ (where $T_{IV}^{ind} < T_{II}^{comb}$)

If so (or, in case (12) was the combination of the third and second spheres and *i* was computed from it), the solution will be: $p/q = 17/4$, $x = 360$ days, $y = 84\frac{12}{17}$ days, and $a = 26^0\,29'$. Because the positions of the sides (*a* and *b*) can be interchanged (cf. (17)), it may have been that $i = 26^0\,29'$, which would make a fairly good approximation to Mercury's true maximum longation ($e = 27^0\,45'$; for Venus $e = 47^0\,30'$). It must be remembered, however, that due to the eccentricities of orbits, the elongations vary. In Mercury ($17^0\,50' \leq e \leq 27^0\,45'$) and in Venus ($45.5^0 \leq e \leq 47.5^0$). In ancient times the elongations were generally given as $e = 20^0$ N. or S. for Mercury and $e = 50^0$ E. or W. for Venus (cf. Chalcidius, *Comment.* § 70, p. 138, ed. Wrobel). The fact that Mercury's elongation was given North or South, may indicate a peculiarity in the treatment of Mercury, like the one suggested above. But even $i = 42^0\,05'$ is possible, for Hipparchus finds errors up to 23^0 (and in one case nearly 60^0) in Eudoxus' description of the colures (*Comm. in Arat.* i, 11, 9-21; ii, 11, 21; cf. [2], p. 155).

M a r s The proportion is

(27) $$\frac{xy}{x+y} = \frac{260\,n}{n}$$

17. Whatever arrangement we accept for the computation of the inclination *i* for Mercury and Venus, we must compare their directions of rotations to those of the Sun. For in the *Timaeus* Plato speaks about motions in opposite directions. Tm. 36 d may be a very concise summary of the motions of the seven sets of planetary spheres in Eudoxus. But at 38d the motions of Venus and Mercury are contrasted with the motion of the Sun. Three possibilities may be discerned here. (a) If the *i* was computed from (12) or (26) for Mercury, it possesses a *dynamis* contrary to the *dynameis* of Venus and the Sun,viz. the power (whatever it was) causing the eastward combined motion of the fourth sphere, in contradistinction to the westward combined motions of the innermost spheres of Venus and the Sun. (b) If the *i* was computed from (12) or (26) for Venus, too ($i = 63^0$, which is still possible; see the note on Eudoxus' observational accuracy in the description of the colures), both Venus and Mercury possess this contrary *dynamis*.(c) The simplest explanation is, however, that since all three have the same period (ibid.) of one year in the zodiacal motion, Venus and Mercury are contrasted with the Sun on the account that they have four motions in Eudoxus' system. The motions of their third spheres of course are contrary to the Sun's ecliptical (or near-ecliptical) motion and so are their retrograde motions. — For a different explanation see Cornford [1], pp. 105-106, but consult also [9], p. 32. n. Heath [4], pp. 165-169, discusses at some length certain ancient explanations. I must add that Harold Cherniss in a letter to me doubts any reference in the *Timaeus* to Eudoxus' systems. Be that as it may, the only things that I really have deduced from Plato are the *loxotes* and the clue into Pythagorean triples.

where the period T=260 days, instead of the true Martian synodic period T^{Syn}_{Mars} = 780 days, probably is the period of tne actual loop (see the argument in [9] pp. 81-84). Besides, Schiaparelli has shown that if the true synodic period is taken, Mars will have no retrograde motion at all, no matter what value *i* is given (see also Dicks [2], pp. 186-187, and Lasserre [1], pp. 205-206).

The two triangles pertaining to the solution are as follows:

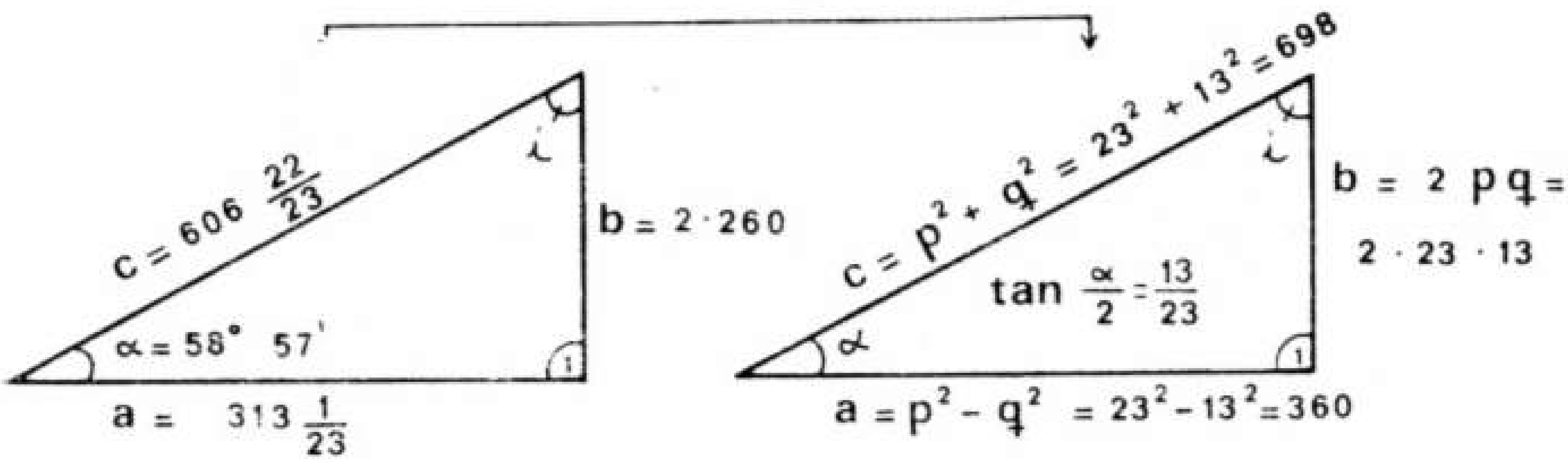


Fig. 8. The two triangles for the fourth motion of Mars.

The solution is: p/q = 23/13, n = $1126\frac{22}{23}$, x = 720 days, y = 406 $\frac{22}{23}$ days, and α = 58° 57′. Hence *i* = 31° 03′ (Schiapaɪelli gave *i* = 34°, but in all outer planets his inclinations are pure guessing, guided by modern spherical trigonometry, for the outer planets may be at any angular distance from the Sun). (If T^{comb}_{IV} = 780 days and the solution comes from (26), i = 35° 09′, or p/q = 25/13).

J u p i t e r The proportion is

$$(28)\qquad \frac{xy}{x+y} = \frac{390\ n}{n}$$

The two triangles pertaining to the solution are as follows.

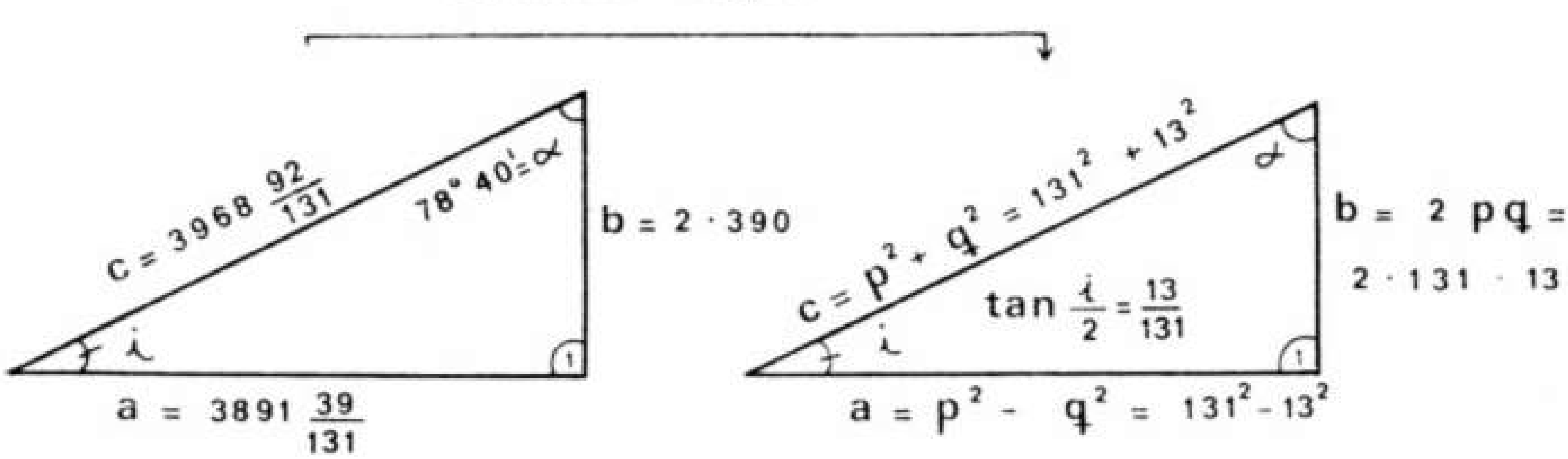


Fig. 9. The two triangles for the fourth motion of Jupiter.

The solution is: $p/q = 131/13$, $n = 12.360.144/131$, $x = 12.360$ days, $y = 428\frac{92}{131}$ days, and $\alpha = 78^0 40'$. Hence $i = 11^0 20'$ (Schiaparelli gave $i = 13^0$).

S a t u r n The proportion is

$$(29) \quad \frac{xy}{x+y} = \frac{390\, n}{n}$$

The two triangles pertaining to the solution are as follows.

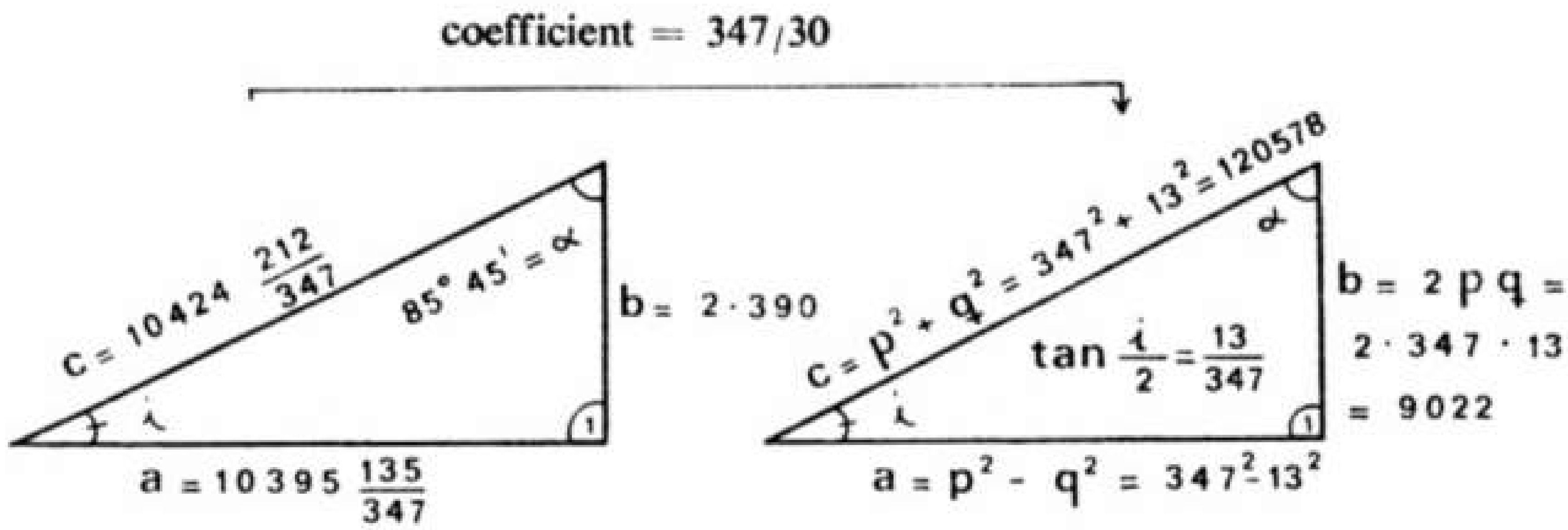


Fig. 10. The two triangles for the fourth motion of Saturn.

Lest the somewhat cumbersome figures seem suspicious, it may be advisable to do the computations necessary in detail, taking Saturn as an example. From (15) we obtain $x = 30 \,.\, 360$ days $= \frac{a+b+c}{2}$, and $b = 2.390$ days.

Hence

$$\begin{array}{r} a+b+c = 60 \,.\, 360 \\ b \quad = 60 \,.\, 13 \\ \hline a+c = 60 \,.\, 347 \end{array}$$

Now $p/q = \frac{a+c}{b} = 347/13$ (a paraphrase for $i = 4^0 15'$; Schiaparelli gave $i = 6^0$). And $y = q/p \,.\, x = 13/347 \,.\, 30 \,.\, 360 = 390/347 \,.\, 360$ days ($\tan \frac{i}{2} = y/x = q/p$). This is all.

7. The third motions of planets proper.

It is, in my opinion, a master-stroke of methodological economy that the combinations of the second and third motions of the planets proper all are of the type (12) which has not been used elsewhere. And even if (26) was used for the combination of the second and fourth spheres of Venus and Mercury, (10) and (12) will suffice for the third motions of the planets proper. If they were studied in their ecliptical projections, as one may well assume, all projections equal zero and the particular solutions are superfluous. If they were studied in equatorial projections, the northward motions mentioned in the tradition will appear as having a direction opposite to the fourth motions (since *hippopedes* are formed).

8. The hippopede constructions.

Schiaparelli's conjectural hippopedes have been studied in detail by several generations of ancient and modern commentators. In the Bibliography of [9] I give the titles of the most representative investigations, and it is unnecessary to dwell on the topic here. Suffice it to say that Schiaparelli's hippopedes are close enough to mine to give an approximative summary. But it must be emphasized that although his inclinations (i) are rather near to mine, mine are implied by Eudoxus' reconstructed method, and do not ensue from modern observation and mathematics (cf. Neugebauer [12]).

9. Conclusion.

I have tried to show that all astronomical parameters, known either accurately or in principle from the Eudoxan tradition, and characterizing the Eudoxan theory of the homocentric spheres, can be obtained as soon as means are discovered for dealing with a generalized proportion of the following type:

$$(30) \quad xy : (x \pm y) = n\,T^{comb} : n$$

Here x,y are two planetary periods (of which one is known or postulated in advance), n is a positive integer or a positive rational fraction needed in the generalization (cf. *Eucl.* v. 15) and T^{comb} is a planet's synodic or sidereal period, known in advance from observation. The main body of my paper consists in the reconstruction of Eudoxus' method, which, when applied to the observational planetary data («the phenomena to be saved»), produces the astronomical parameter values. These include both real, observational values (of which some are rather accurate) and entirely fictitious ones. The success of the reconstruction may be judged from the fact that also the fictitious values, such as the inclination of the Sun's third sphere, can be obtained.

The main types of parameter values obtained are the planetary periods (e.g. T_{II}^{ind} = the period of the individual motion of a planet's second sphere, T_{II}^{comb} = the period of the combined motion of a planet's second sphere, etc.), the directions of the spherical rotations (e.g., $D_{IV}^{ind(W)}$ = the westward individual motion of a planet's fourth sphere, etc.), the inclinations of the spherical axes and, being deducible from these, the maximum deviations of a planet's sphere from the equator (including the Eudoxan value for the obliquity of the ecliptic) and from the ecliptic.

In essence, the reconstructed Eudoxan method is based on Pythagorean mathematics, and pains have been taken to show that it can be backed by propositions belonging to Eudoxus' known contribution to the *Elementa*. Eudoxus' method consists of two parts: an a n a l y s i s and a s y n t h e s i s. Because Eudoxus' importance is acknowledged by almost all subsequent promotors of mathematical analysis (for more detail see my paper *The Elements of Analysis*, «Proceedings of the XIV International Congress of the History of Science», 1974, Tokyo and Kyoto), and because Eudoxus' method has affected much contemporary philosophical analysis and synthesis, my reconstruction may be of some interest to the historian and philosopher of the exact sciences. Besides, it may illustrate the logic of scientific discovery.

At the heart of the reconstructed method (discussed from the algebraic point of view in the Appendix) lies in each case a Pythagorean triangle, different for each planet. Its sides are generated from two relatively prime, unequal integers p, q in the usual way, and the triangle itself has the role of an auxiliary drawing in a geometrical proof. In terms of this triangle both the auxiliary parameter *n* and the solution to (30) can be given, and its angles determine the axial inclinations and maximum deviations. The solutions to (30) are invariable with respect to p, q (which may be called «the mathematical constants of nature») and can be obtained in each case in the form: $x/y = p/q$. The solution gives rise to the tangent (= ratio of the *gnomon* to its shadow) of the angles of inclination, say *i*, and maximum deviation, say α. In fact, it also suggests the main features of the astronomical instrument (*arachne*) which Eudoxus has used in measuring angular distances. Moreover, the inverse of $p/q = \tan\frac{\alpha}{2}$ or $= \tan\frac{i}{2}$ (cf. *Eucl.* III. 18, VI. 3), which leads to the theory of stereographic projection, expedient in the calculation of angular velocities, paraphrased in termes of periods in (30). These further implications of the reconstruction, however, will be discussed elsewhere. Suffice it to say that Eudoxus' method can be understood as a geometrical treatment of time, and that his main heuristic aid has been the Pythagorean numerical analysis (cf. Iambl., *In Nicom. arithm.* 10, 17).

I list here the most important parameter values explained. (D = direction of rotation).

T h e M o o n : $T_{II}^{comb} = 30$ days, $D_{III}^{ind(W)}$, $T_{III}^{ind} = 870$ days = the «long» lunar period, $T_{III}^{comb} = 29$ days = the «shorter» period = the «hollow» month, and $\alpha = 3^0 57'$ $(p/q = 29/1)$, $y/x = q/p = 1/29 = \tan(\alpha/2)$.

T h e S u n : $T_{II}^{comb} = 360$ days, $D_{III}^{ind(E)}$, $T_{III}^{ind} = 32760$ days = the «long» solar period, T^{comb}_{III} = 364 days = the «shorter» period = the «seasonal year» 4 . 91 days, and α = 1°15.5′ = appr. half an e l l; (p/q = 91/1).

V e n u s: T^{comb}_{II} = 360 days, $D^{ind(E)}_{IV}$, incl.$_{III}$ = the same as for Mercury and the Sun, while for other planets this will be different for each, T^{comb}_{III} = T^{comb}_{IV} = 570 days, and incl.$_{IV}$ = 49°33′ (p/q = 19/7) . (Note also the other alternatives discussed above).

M e r c u r y: T^{comb}_{II} = 360 days, $D^{ind(W)}_{IV}$, T^{comb}_{III} = T^{comb}_{IV} = $110\frac{10}{13}$ days, and incl.$_{IV}$ = 42°05′ (p/q = 9/4). (Note also the other alternatives discussed above).

M a r s: T^{comb}_{II} = 2 . 360 days, $D^{ind(W)}_{IV}$, T^{comb}_{III} = T^{comb}_{IV} = 260 days (which is the period of the actual loop, while T^{syn} = 780 days), and incl.$_{IV}$ = = 31° 03′ (p/q = 23/13). (Note also the alternative T^{comb}_{III} = 260, T^{comb}_{IV} = 780 days, i = 35° 09′).

J u p i t e r: T^{comb}_{II} = 12 . 360 days, $D^{ind(W)}_{IV}$, T^{comb}_{III} = T^{comb}_{IV} = 390 days, and incl.$_{IV}$ = 11° 20′ (p/q = 131/13).

S a t u r n: T^{comb}_{II} = 30 . 360 days, $D^{ind(W)}_{IV}$, T^{comb}_{III} = T^{comb}_{IV} = 390 days, and incl.$_{IV}$ = 4°17′ (p/q = 347/13).

L o x o t e s: a = 22°37′ (p/q = 3/2); the real value in Eudoxus' day ε = = 23° 44′.

It will be seen that calendaric considerations indeed played a role in Eudoxus' concept of empirical data, and that the *hippopede* constructions really aimed at approximative (or idealized) descriptions of the observed planetary motions. But it must have been something of a shock for Eudoxus and his followers to discover that one and the same method of computation indeed produces parameter values that were well known from the past. It must have appeared as if in (1) a «law of nature» had been invented. A «law» which, albeit simple, nonetheless indicates in an economical interpretation the «constants of nature».

The inner beauty of Eudoxus' system, even in the barest outline drawn in this paper, is gripping. The simplicity of the mathematical tools, the very concrete, yet strikingly effective methods of analysis and synthesis, the extreme parsimony of the e x p l a n a n s and the ingenious interpretation, contrasted with the multitude of the phenomena explained, are to the full credit of Eudoxus' genius. His solution to the cosmological problem is characterized by strong methodological monism, which Philipp of Opus probably refers to in the *Epinomis* 991e-992a. Other details may be omitted here, but suffice it to say that if for instance the sum and difference of p and q in each particular solution are considered, it will be noted that the same numbers (12, 13, 30, 36, 90, and 360) occur at all levels of the explanation, strongly suggesting that one has seen into the depths of the celestial architecture. At the same time, however, Eudoxus' system was capable of further development by his followers. Hence it is not a system of sterile rigidity like Aristotle's, but the beginning of a dynamic paradigm.

Eudoxus continued active studies on Cnidus until the end of his life. Strabo (C 119), following Poseidonius, mentions that he observed Canopus (α Carinae): this is probably connected with his attempt at an estimate of the diameter of the Earth. But the mathematical foundations of the Greek astronomy (and ours) were laid, as cogently as it was possible for Eudoxus, in his theory of the homocentric spheres. If the reconstruction of Eudoxus' method submitted above is correct, the philosophical discussion of such themes as the real meaning of the principle of «saving the phenomena», the concept of time and its mathematical treatment, and the heuristic value of the methods of analysis and synthesis in other, more general problems, may begin.

When viewing Eudoxus' theory of the homocentric spheres as a whole, one notes the striking discrepancy between the geometrical model and the physical reality. Not only is it so that a geometrical framework is insufficient in dealing with the computation of angular velocities, and the results obtained approximative. The model, albeit in motion, is far from the modern ideal of physical models which try to preserve the possibility of checking a t a n y t i m e. Eudoxus' model, on the other hand, «saved» a fixed number of phenomena a t a g i v e n t i m e, and it had predictive value in the case of each planet at a given time only, viz. at the time when a planet's maximum deviation was observable. With respect to these «saved phenomena», however, the model is in full correspondence with reality. That is to say, the language of Eudoxus' theory exhibits the idea of logical or semantical atomism at certain fixed times.

One can understand this view of language against the contemporary semantical theories in Greek philosophy (see e.g. my paper *The Semantics of Time in Plato's Timaeus,* «Acta Academiae Aboensis», vol. 38, No. 3, 1970). For the Pythagoreans «the whole universe is filled with numbers» (cf. Ar. *Met.*, 1090a 20 sq.), but for Aristotle already the contents of the two boxes, one containing the language and linguistic models, the other containing reality, are different. We can almost put our finger on the point of divergence: at *De Caelo* 289b Aristotle discusses the possibility of motions of stars and their circles independently from each other. Hitherto, a planet and its sphere had formed a unity, one physical body if we wish to put it so.

But even if we grant this somewhat alien idea of physical bodies, the fact remains that Eudoxus' model was in full correspondence with reality at given times only. For surely no physical body can rotate on three or four spheres of different radii at the same time. And different radii they had, as Aristotle clearly implies (notwithstanding what modern commentators have claimed), for that is essential if stereographic projections are used in the calculation of angular velocities. Nevertheless, the «temporary» agreement of Eudoxus' model with the «saved phenomena», as well as his strikingly concrete method of analysis (or should we say, in view of A'rpâd Szabó's results concerning the origins of terms for «proof» and «showing», Eudoxus' optical method of analysis), are extremely effective theoretical tools.

It is with great delight, therefore, that I can end this paper by pointing out that the attitude towards models and their agreement with reality was exactly of the type required in Plato's Academy (cf. Ar. *Met.* 990a). For my working hypothesis even in this paper has been that Plato in the *Timaeus* is using Eudoxus' theory as the frame of reference for his own fragmentary astronomy. It is from the *Timaeus* that I have discovered the clues to Eudoxus' Pythagorean triples and Eudoxus' obliquity of the ecliptic (and Plato's *great harmonia* provides the *arachne* with two scales, musical and geometrical harmonic scales for angular measurement). Now, at *Tm.* 37c Plato uses the term *agalma* for the relation between the World-Soul and the planets, and I have shown elsewhere (*Plato's Agalma of the Eternal Gods,* «Yearbook of the Philosophical Society of Finland», 1969) that agalma in Plato is a pregnant philosophical metaphor, exceptionally apt for the description of the relationship between «everlasting» and «temporal» beings. In Plato's hands, *agalma* has undergone a change from a religious concept into a philosophical one. But it has preserved some of the original religious connotations. The agalma at *Tm.* 37c is (i) a likeness of its paradigm and the Demiurge tries to make it «yet more like its pattern», (ii) it is «set in motion and alive», (iii) it is the object of the Demiurge's delight, and (iv) it seems to have a function similar to cult-statues of which the divinities are supposed to discuss — at least during the exalted moments of actual worship.

It is in this intellectual atmosphere that Eudoxus presented a geometrical model that «saved the phenomena», but fully corresponded to reality at given times only. Like so many later mathematicians, Eudoxus has created a visible model, the heuristic value of which lies precisely in the fact that whatever is done with the model will suggest particular solutions to problems discovered in reality. Is this not the same perennial ideal that also philosophers like Leibniz and Wittgenstein have aimed at with their theories anchored to semantical atomism?

APPENDIX*

An algebraic view.

In Eudoxus' cosmological construction, angular velocities (inverses of periods) can be combined by projecting them pairwise from one sphere to another. From the mathematical point of view, such combinations can be expressed by the following formula:

$$(1) \quad \frac{1}{T_1} \pm \frac{1}{T_2} = \frac{T_1 \pm T_2}{T_2\,T_2} = \frac{1}{T^{comb}}$$

In order to master the combined motion Eudoxus thus had to be capable of solving the algebraic problem: Given T^{comb} give x_1 and x_2 such that

$$(2) \quad \frac{x_1 \,.\, x_2}{x_1 \pm x_2} = T^{comb}$$

With the known *mathematical tools* of that age the problem was unsolvable. Hence we try to find out a possible reconstruction by the analogy of which Eudoxus could have solved (2) using the *methods of analysis and synthesis*. Although the following algebraic formalism is perhaps anachronistic, contemporary mathematicians, however, would have been capable of making essentially the same reasoning.

To begin with let us consent to Erkka Maula's working hypothesis that Eudoxus knew the (far older) Babylonian method of normal forms to solve quadratic equations.

$$(3) \text{ problem} \begin{cases} x_1 \,.\, x_2 = a \\ x_1 \pm x_2 = b \end{cases} \quad \text{solution} \begin{cases} x_1 = \frac{b}{2} + \sqrt{\left(\frac{b}{2}\right)^2 \mp a} \\ x_2 = \pm \frac{b}{2} \mp \sqrt{\left(\frac{b}{2}\right)^2 \mp a} \end{cases}$$

When looking for the characteristic features of (2) in the *analysis* we realise that there are in fact the equated ratios of left and right sides of the normal forms.

$$(4) \begin{cases} x_1 x_2 = T^{comb} \\ x_1 \pm x_2 = 1 \end{cases} \quad \left[\text{solution} \begin{cases} x_1 = 1/2 + \sqrt{1/4 \mp T^{comb}} \\ x_2 = \pm 1/2 \mp \sqrt{1/4 \mp T^{comb}} \end{cases} \right]$$

After this observation it is natural to examine the solution of (4) — and to our pleasure it satisfies the original problem (2). Heuristic «optical» partition yielded a fruitful result — let us continue in the same manner. Problem (2) has an infinite set of solutions and basically (2) is a ratio. How can pro-

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blem (4) be generalized accordingly? We introduce a parameter *n*. Let $x_1 \pm x_2$ be equal to n, then $x_1 \cdot x_2 = T^{comb} \cdot n$. Thus the characteristic features of the original problem have been maintained. Eudoxus' assumed contribution to *Elementa* includes

(5) $$\frac{a}{b} = \frac{an}{bn} \qquad (v \,.\, 15)$$

which gives evidence of the preceding mode of reasoning. Problem (2) is altered into an equivalent form $\frac{x_1 \cdot x_2}{x_1 \pm x_2} = \frac{T^{comb} \cdot n}{n}$ and so the respective normal forms acquired will be

(6) $$\begin{cases} x_1 \cdot x_2 = T^{comb} \cdot n \\ x_1 \pm x_2 = n \end{cases} \qquad \text{solution} \begin{cases} x_1 = \frac{n}{2} + \sqrt{\left(\frac{n}{2}\right)^2 \mp T^{comb} \cdot n} \\ x_2 = \pm\frac{n}{2} \mp \sqrt{\left(\frac{n}{2}\right)^2 \mp T^{comb} \cdot n} \end{cases}$$

When testing this result the following conclusion can be immediately drawn: allowing parameter *n* to take several values an infinite set of pairs x_1, x_2 satisfying (2) is obtained. The only flaw is the permanent presence of parameter *n* which logically corresponds to auxiliary designs in the geometrical analysis.

We proceed to the s y n t h e s i s restoring the ratio split in the analysis. Partitioning is compensated for by building the ratio anew. The solutions x_1, x_2 in (6) are moulded into the ratio $x_1 : x_2$ which is called the s o l u t i o n of the original problem. Calculating

(7) $$\frac{x_1}{x_2} = \frac{\frac{n}{2} + \sqrt{\left(\frac{n}{2}\right)^2 \mp T^{comb} \cdot n}}{\pm\frac{n}{2} \mp \sqrt{\left(\frac{n}{2}\right)^2 \mp T^{comb} \cdot n}} = \frac{\left(\frac{n}{2} + \sqrt{\left(\frac{n}{2}\right)^2 \mp T^{comb} \cdot n}\right) \mp T^{comb}}{T^{comb}} = \frac{x_1 \mp T^{comb}}{T^{comb}}$$

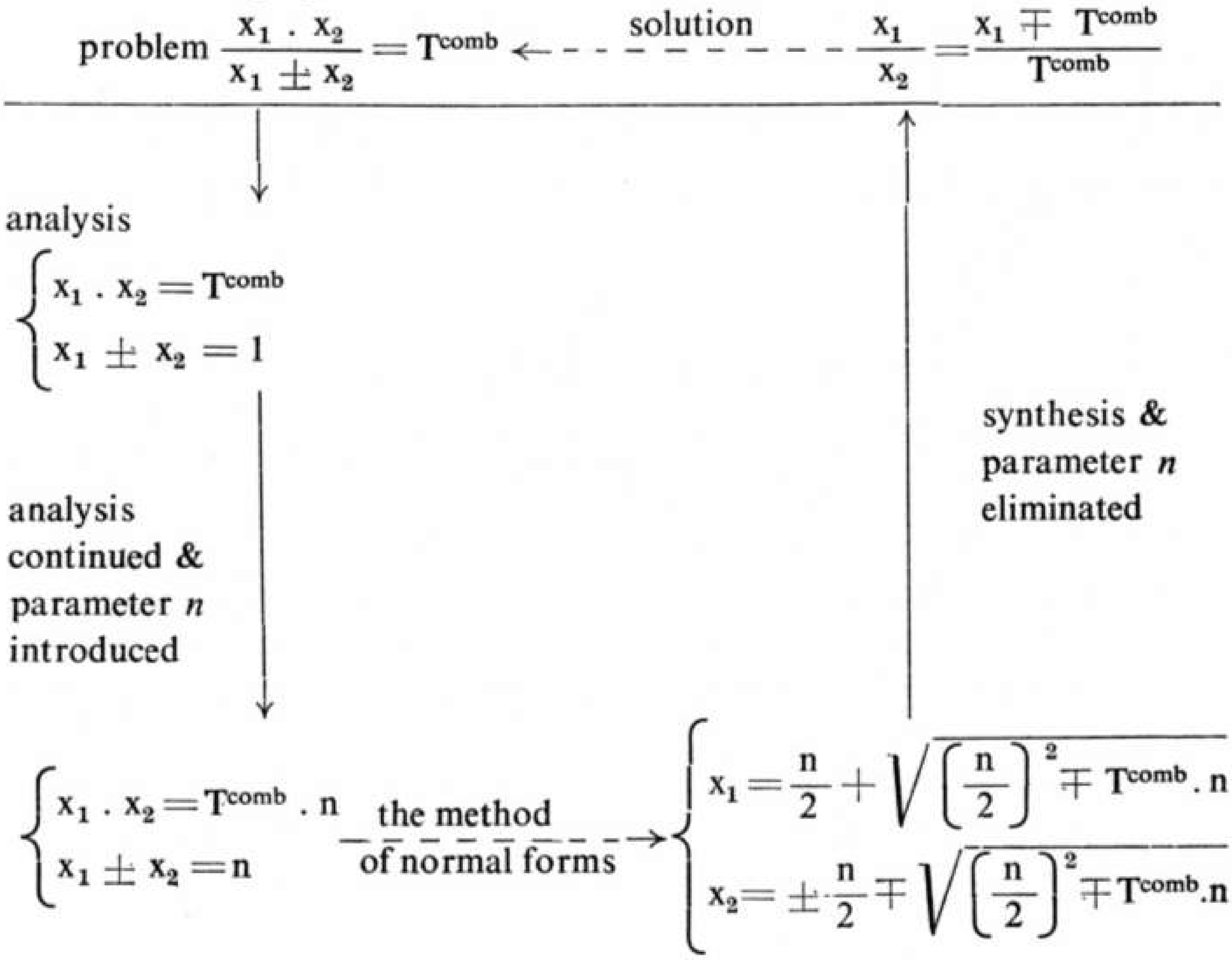
we witness the striking disappearance of parameter n. All pairs of number x_1, x_2 satisfying the condition

(8) $$\frac{x_1}{x_2} = \frac{x_1 \mp T^{comb}}{T^{comb}}$$

also satisfy the original problem. The value of parameter $n = x_1 \pm x_2$ is seen in (6).

What have we gained? The problem was divided into parts by using the above outlined method of analysis. Parameter *n* was introduced to maintain the characteristic features. The pair of simultaneous equations was solved by the method of normal forms. The solution was defined and calculated in the synthesis in which parameter *n* was also eliminated.

Let the following figure illustrate the situation.



ΟΙ ΣΤΑΘΕΡΕΣ ΤΗΣ ΦΥΣΕΩΣ

ΜΙΑ ΜΕΛΕΤΗ ΓΙΑ ΤΗΝ ΠΡΩΙΜΗ ΙΣΤΟΡΙΑ ΤΟΥ ΦΥΣΙΚΟΥ ΝΟΜΟΥ

Περίληψη.*

Στὴ μελέτη αὐτὴ γίνεται προσπάθεια νὰ δειχθῆ, ὅτι ὅλες οἱ ἀστρονομικὲς παράμετροι, ποὺ εἶναι γνωστὲς εἴτε ἐπακριβῶς εἴτε κατ᾽ ἀρχὴν ἀπὸ τὴν παράδοση τοῦ Εὐδόξου καὶ ποὺ χαρακτηρίζουν τὴ θεωρία του γιὰ τὶς ὁμόκεντρες σφαῖρες, μποροῦν νὰ ἐξαχθοῦν ὅταν βρεθῆ τρόπος νὰ μελετηθοῦν μὲ μιὰ γενικευμένη ἀναλογία τοῦ τύπου:

$$(30) \quad xy : (x \pm y) = n\, T^{comb} : n.$$

* Μετάφραση ἀπὸ τὸ ἀγγλικὸ Μ. Δραγώνα-Μονάχου.

Ἐδῶ x,y εἶναι δύο πλανητικὲς περίοδοι (ἀπὸ τὶς ὁποῖες ἡ μία εἶναι γνωστὴ ἢ παραδεδεγμένη ὡς ἀξίωμα ἐκ τῶν προτέρων), *n* εἶναι θετικὸς ἀκέραιος ἢ θετικὸς ρητὸς κλασματικὸς ἀριθμός, ἀπαραίτητος στὴ γενίκευση (πρβλ. Εὐκλ. V - 15), καὶ T^{comb} εἶναι ἡ συνοδικὴ ἢ ἀστρικὴ περίοδος ἑνὸς πλανήτη, γνωστὴ προκαταβολικῶς ἀπὸ τὴν παρατήρηση.

Τὸ κύριο μέρος τοῦ ἄρθρου πραγματεύεται τὴν ἀνακατασκευὴ τῆς μεθόδου τοῦ Εὐδόξου, ἡ ὁποία, ὅταν ἐφαρμοσθῆ στὰ πλανητικὰ ἀπὸ παρατηρήσεις δεδομένα («γιὰ νὰ σωθοῦν τὰ φαινόμενα»), δίνει τὶς ἀστρονομικὲς τιμὲς τῶν παραμέτρων. Σ' αὐτὲς περιλαμβάνονται καὶ πραγματικές, ἐμπειρικὲς τιμὲς (μερικὲς ἀπὸ τὶς ὁποῖες εἶναι μᾶλλον ἀκριβεῖς καὶ ἄλλες τελείως πλασματικές). Ἡ ἐπιτυχία τῆς ἀνακατασκευῆς μπορεῖ νὰ κριθῆ ἀπὸ τὸ ὅτι μποροῦν νὰ ἐξαχθοῦν ἀκόμα καὶ οἱ πλασματικὲς τιμές, ὅπως π.χ. ἡ ἀπόκλιση τῆς τρίτης σφαίρας τοῦ Ἡλίου.

Οἱ κύριοι τύποι τιμῶν παραμέτρων ποὺ ἐξήχθησαν εἶναι οἱ πλανητικὲς περίοδοι (π.χ. T_{II}^{ind} = ἡ περίοδος τῆς ἀτομικῆς κινήσεως τῆς δεύτερης σφαίρας ἑνὸς πλανήτη, T_{II}^{comb} = ἡ περίοδος συνδυασμένης κινήσεως τῆς δεύτερης σφαίρας ἑνὸς πλανήτη κλπ.), οἱ φορὲς τῶν σφαιρικῶν περιστροφῶν (π.χ. $D_{IV}^{ind\,(W)}$ = ἡ ἀτομικὴ κίνηση τῆς τέταρτης σφαίρας ἑνὸς πλανήτη πρὸς δυσμάς, κλπ.), οἱ ἀποκλίσεις τῶν σφαιρικῶν ἀξόνων καί, αὐτὸ ποὺ μπορεῖ νὰ συναχθῆ ἀπ' αὐτές, οἱ μέγιστες παρεκκλίσεις τῆς σφαίρας ἑνὸς πλανήτη ἀπὸ τὸν ἰσημερινὸ (ὅπου περιλαμβάνεται καὶ ἡ τιμὴ τῆς λοξότητας τῆς ἐκλειπτικῆς τοῦ Εὐδόξου) καὶ ἀπὸ τὴν ἐκλειπτική.

Ἡ μέθοδος τοῦ Εὐδόξου ποὺ ἀνακατασκευάσαμε βασίζεται οὐσιαστικὰ στὰ πυθαγορικὰ Μαθηματικά· καταβάλαμε προσπάθειες νὰ δείξωμε ὅτι μπορεῖ νὰ ὑποστηριχθῆ μὲ προτάσεις, ποὺ ἀνήκουν στὴ γνωστὴ συμβολὴ τοῦ Εὐδόξου στὰ *Στοιχεῖα*. Ἡ μέθοδος ἀποτελεῖται ἀπὸ δύο μέρη: μιὰ ἀνάλυση καὶ μιὰ σύνθεση. Ἐπειδὴ ἡ σημασία τοῦ Εὐδόξου ἀναγνωρίζεται ἀπὸ ὅλους σχεδὸν τοὺς μεταγενέστερους ὀπαδοὺς τῆς μαθηματικῆς ἀνάλυσης (γιὰ περισσότερες λεπτομέρειες βλέπε τὸ ἄρθρο μου *The Elements of Analysis,* «Proceedings of the XIV International Congress of the History of Science», Tokyo /Kyoto, 1974), καὶ ἐπειδὴ ἡ μέθοδος τοῦ Εὐδόξου ἐπηρέασε τὴ σύγχρονη φιλοσοφικὴ ἀνάλυση καὶ σύνθεση σὲ μεγάλη ἔκταση, ἡ ἀνακατασκευὴ μπορεῖ νὰ παρουσιάζη κάποιο ἐνδιαφέρον γιὰ τὸν ἱστορικὸ καὶ τὸ φιλόσοφο τῶν θετικῶν ἐπιστημῶν. Ἀκόμα, μπορεῖ νὰ διαφωτίση τὴ λογικὴ τῆς ἐπιστημονικῆς ἀνακάλυψης.

Στὸ κέντρο τῆς μεθόδου ποὺ ἀνακατασκευάσαμε (καὶ ποὺ ἀπὸ τὴν ἀλγεβρικὴ ἄποψη συζητεῖται στὸ παράρτημα τοῦ ἄρθρου) βρίσκεται ἕνα πυθαγορικὸ τρίγωνο γιὰ κάθε περίπτωση, διαφορετικὸ γιὰ κάθε πλανήτη. Οἱ πλευρές του παράγονται ἀπὸ δύο σχετικῶς πρώτους ἀνίσους ἀκεραίους ἀριθμοὺς p, q μὲ τὸ συνηθισμένο τρόπο, καὶ τὸ ἴδιο τὸ τρίγωνο ἔχει τὸ ρόλο βοηθητικοῦ σχεδίου σὲ μιὰ γεωμετρικὴ ἀπόδειξη. Τόσο ἡ βοηθητικὴ παράμετρος n ὅσο καὶ ἡ λύση στὸν τύπο (30) μποροῦν νὰ δοθοῦν σὲ συνάρτηση πρὸς τὸ τρίγωνο αὐτό, καὶ οἱ γωνίες του καθορίζουν τὶς ἀξονικὲς ἀποκλίσεις καὶ τὶς μέγιστες παρεκκλίσεις. Οἱ λύσεις στὸ (30) εἶναι ἀμετάβλητες ἀναφορικὰ πρὸς τὸ p, q (τὰ ὁποῖα μποροῦν νὰ ὀνομασθοῦν «οἱ μαθηματικὲς σταθερὲς τῆς φύσεως») καὶ μποροῦν νὰ ἐξαχθοῦν μὲ τὸν τύπο: $x/y = p/q$ μὲ κάθε περίπτωση. Ἡ λύση συνεπάγεται τὴν ἐφαπτομένη (= ἀναλογία τοῦ *γνώμονος* πρὸς τὴ σκιά του) τῶν γωνιῶν ἀπόκλισης, π.χ. i καὶ μέγιστη (τιμὴ) παρέκκλισης π.χ. α. Ὑποδηλώνει ἀκόμα, στὴν πραγματικότητα, τὰ κύρια χαρακτηριστικὰ τοῦ ἀστρονομικοῦ ὀργάνου (*ἀράχνη*), ποὺ χρησιμοποίησε ὁ Εὔδοξος γιὰ νὰ μετρήσῃ γωνιακὲς ἀποστάσεις, κι ἀκόμη τὸ ἀντίστροφο τοῦ $p/q = \tan\frac{\alpha}{2}$ ἢ $= \tan\frac{i}{2}$ (βλ. Εὐκλ. III 18, VI 3), ποὺ ὁδηγεῖ στὴ θεωρία τῆς στερεογραφικῆς προβολῆς, πρόσφορη γιὰ τὸν ὑπολογισμὸ γωνιακῶν ταχυτήτων, ποὺ παραφράζονται σὲ συνάρτηση πρὸς τὶς περιόδους στὸ (30). Ὡστόσο οἱ περαιτέρω αὐτὲς συνέπειες τῆς ἀνασύνθεσης θὰ συζητηθοῦν ἀλλοῦ. Ἀρκεῖ πρὸς τὸ παρὸν νὰ ποῦμε ὅτι ἡ μέθοδος τοῦ Εὐδόξου μπορεῖ νὰ νοηθῇ ὡς γεωμετρικὴ ἑρμηνεία τοῦ χρόνου καὶ ὅτι ἡ κύρια «εὑρετικὴ» βοήθειά του ὑπῆρξε ἡ πυθαγόρεια ἀνάλυση τῶν ἀριθμῶν.

Ὁ πίνακας τῶν σημαντικωτέρων παραμετρικῶν τιμῶν ἐξηγεῖται στὸ κεφ. 9. Θὰ φανῇ ἔτσι ὅτι πράγματι ἡμερολογιακὰ ζητήματα ἔπαιξαν ἕνα ρόλο στὴ θεωρία τῶν ἐμπειρικῶν δεδομένων τοῦ Εὐδόξου καὶ ὅτι οἱ κατασκευὲς μὲ βάση τὴν *ἱπποπέδη* εἶχαν στὴν πραγματικότητα σκοπὸ νὰ περιγράψουν κατὰ προσέγγιση (ἢ ἐξιδανικευμένα) τὶς προσιτὲς στὴν παρατήρηση πλανητικὲς κινήσεις. Πρέπει ὅμως νὰ ἦταν κάπως συγκλονιστικὸ γιὰ τὸν Εὔδοξο καὶ τοὺς ὀπαδούς του νὰ ἀνακαλύψουν ὅτι μία καὶ ἡ αὐτὴ μέθοδος ὑπολογισμοῦ ἐξάγει πράγματι παραμετρικὲς τιμὲς ποὺ ἦταν ἀρκετὰ γνωστὲς ἀπὸ τὸ παρελθόν. Καὶ ὁ τύπος (1) πρέπει νὰ φάνηκε ὡς ἀνακάλυψη ἑνὸς νόμου τῆς φύσεως, ἑνὸς νόμου, ποὺ ἂν καὶ ἁπλὸς δείχνει ἐν τούτοις μὲ μιὰ «οἰκονομικὴ» ἑρμηνεία τὶς σταθερὲς τῆς φύσεως. Τὸ ἐσωτερικὸ κάλλος τοῦ συστήματος τοῦ Εὐδόξου, ἐντυπωσιακὸ ἀκόμη καὶ στὴν ἁδρομερῆ σκιαγραφία ποὺ ἐπιχειρήσαμε στὸ ἄρθρο αὐτό, ἡ ἁπλότητα τῶν μαθηματικῶν ἐργαλείων, ἡ ὑπερβολικὴ φειδὼ τῶν ἐξηγητέων ὅρων καὶ ἡ μεγαλοφυὴς ἑρμηνεία, σὲ ἀντιδιαστολὴ μὲ τὸ πλῆθος τῶν φαινομένων ποὺ ἐξηγεῖ, πρέπει ἀνεπιφύλακτα νὰ ἐγγραφῇ στὸ ἐνεργητικὸ τῆς μεγαλοφυίας τοῦ Εὐδόξου. Ἡ λύση του στὸ κοσμολογικὸ πρόβλημα χαρακτηρίζεται ἀπὸ ἕνα ἰσχυρὸ μεθοδολογικὸ μονισμό, στὸν ὁποῖο πιθανῶς ἀναφέρεται ὁ Φίλιππος ὁ Ὀπούντιος στὴν *Ἐπινομίδα* (991 e-992 e).

Ὁ Εὔδοξος ἐξακολούθησε ἐνεργῶς τὶς μελέτες του στὴν Κνίδο ὣς τὸ τέλος τῆς ζωῆς του. Ὁ Στράβων (c 119), ἀκολουθώντας τὸν Ποσειδώνιο, ἀναφέρει ὅτι παρατήρησε τὴν Κάνωπο (α Carinae). Τοῦτο προφανῶς συνδέεται μὲ τὴν προσπάθειά του νὰ ὑπολογίση τὴ διάμετρο τῆς Γῆς. Ἀλλὰ τὰ μαθηματικὰ θεμέλια τῆς Ἑλληνικῆς Ἀστρονομίας (καὶ τῆς ἰδικῆς μας) ἐτέθησαν ὅσο πιὸ ἰσχυρὰ γινόταν γιὰ τὸν Εὔδοξο μὲ τὴ θεωρία του τῶν ὁμοκέντρων σφαιρῶν.

Ὅταν θεωρήση κανεὶς τὴ θεωρία τῶν ὁμοκέντρων σφαιρῶν τοῦ Εὐδόξου ὡς σύνολο, παρατηρεῖ τὴν ἐντυπωσιακὴ διαφορὰ τοῦ γεωμετρικοῦ προτύπου ἀπὸ τὴ φυσικὴ πραγματικότητα. Ὄχι μόνο αὐτὸ ἀλλὰ καὶ τὸ γεωμετρικὸ πλαίσιο εἶναι ἀνεπαρκές, ὅταν ἔχη νὰ κάνη κανεὶς μὲ τὸν ὑπολογισμὸ γωνιακῶν ταχυτήτων καὶ μὲ τὰ κατὰ προσέγγιση ἐπιτευχθέντα ἀποτελέσματα. Τὸ πρότυπο, ἂν καὶ κινηματικό, ἀπέχει πολὺ ἀπὸ τὸ νεώτερο ἰδανικὸ τῶν φυσικῶν προτύπων, ποὺ προσπαθοῦν νὰ διατηρήσουν τὴ δυνατότητα ἐπαλήθευσης (ἐλέγχου) σὲ ὁποιονδήποτε χρόνο. Τὸ πρότυπο τοῦ Εὐδόξου ὅμως «ἔσωσε» ἕνα σταθερὸ ἀριθμὸ φαινομένων σὲ δεδομένο χρόνο καὶ εἶχε προφητικὴ ἀξία στὴν περίπτωση κάθε πλανήτη μόνο σὲ δεδομένο χρόνο, δηλαδὴ στὸ χρονικὸ σημεῖο ὅπου ἡ μεγίστη παρέκκλιση ἑνὸς πλανήτη ἦταν δυνατὸν νὰ παρατηρηθῆ. Πάντως ἀναφορικὰ μὲ αὐτὰ τὰ «σωθέντα φαινόμενα» τὸ πρότυπο ἀνταποκρίνεται πλήρως πρὸς τὴν πραγματικότητα. Ἡ γλῶσσα τῆς θεωρίας τοῦ Εὐδόξου ἐκθέτει τὴν ἰδέα τοῦ λογικοῦ ἢ «σημαντικοῦ» ἀτομισμοῦ σὲ ὡρισμένους σταθεροὺς χρόνους.

Μπορεῖ κανεὶς νὰ καταλάβη τὴν ἄποψη αὐτὴ γιὰ τὴν γλῶσσα σὲ σύγκριση μὲ σύγχρονές της θεωρίες «Σημαντικῆς» στὴν ἑλληνικὴ φιλοσοφία (πρβλ. *The Semantics of Time in Plato's Timaeus,* «Acta Academiae Aboensis», vol. 38, No 3, 1970). Γιὰ τοὺς Πυθαγορείους τὸ ὅλο σύμπαν εἶναι πλῆρες ἀριθμῶν (πρβλ. Ἀριστ. *Μεταφ.* 1090 a 20 ἑπ.), ἀλλὰ ἤδη γιὰ τὸν Ἀριστοτέλη τὰ περιεχόμενα καὶ τῶν δύο «κουτιῶν», ποὺ τὸ ἕνα περιέχει τὴ γλῶσσα καὶ τὰ γλωσσικὰ πρότυπα καὶ τὸ ἄλλο τὴν πραγματικότητα, εἶναι διαφορετικά. Μποροῦμε σχεδὸν νὰ ψηλαφίσωμε τὰ σημεῖα τῆς διαφορᾶς. Στὸ *Περὶ οὐρανοῦ* ὁ Ἀριστοτέλης συζητᾶ τὴ δυνατότητα τῆς κίνησης τῶν ἄστρων καὶ τῆς τροχιᾶς τους ἀνεξάρτητα ἀπὸ τὴ μεταξύ τους σχέση, ἐνῶ μέχρι τότε ἕνας πλανήτης καὶ ἡ σφαῖρα του σχημάτιζαν ἑνότητα, ἕνα φυσικὸ σῶμα, θὰ λέγαμε.

Ἀλλὰ καὶ ἂν ἀκόμα δεχθοῦμε αὐτὴ τὴν κάπως ἄσχετη ἰδέα τῶν φυσικῶν σωμάτων, γεγονὸς παραμένει ὅτι τὸ πρότυπο τοῦ Εὐδόξου βρισκόταν σὲ πλήρη ἀνταπόκριση μὲ τὴν πραγματικότητα μόνο σὲ δεδομένους χρόνους. Γιατὶ βεβαίως κανένα φυσικὸ σῶμα δὲν μπορεῖ νὰ περιστρέφεται πάνω σὲ τρεῖς ἢ τέσσερες σφαῖρες μὲ διαφορετικὲς ἀκτῖνες συγχρόνως. Καὶ εἶχαν πράγματι διαφορετικὲς ἀκτῖνες, ὅπως σαφῶς προϋποθέτει ὁ Ἀριστοτέλης (παρὰ τοὺς διαφορετικοὺς ἰσχυρισμοὺς νεωτέρων σχολιαστῶν), γιατὶ αὐτὸ εἶναι οὐσιῶδες, ὅταν γίνεται χρήση στερεογραφικῶν προβολῶν στὸν ὑπολογισμὸ τῶν γωνιακῶν ταχυτήτων. Ἐν τούτοις ἡ «στιγμιαία» συμφωνία τοῦ προτύπου τοῦ Εὐδόξου μὲ τὰ «σωθέντα φαινόμενα», ὅπως ἐπίσης καὶ ἡ ἐντυπωσιακὰ συγκεκριμένη μέθοδός του ἀνάλυσης (ἢ ἐν ὄψει καὶ τῶν ἀποτελεσμάτων τοῦ A'rpád Szabó ἀναφορικὰ μὲ τὶς ἀπαρχὲς τῶν ὅρων *ἀπόδειξις* καὶ *δεῖξις*, θὰ ἔπρεπε νὰ ποῦμε ἡ ὀπτικὴ μέθοδος ἀνάλυσης τοῦ Εὐδόξου) εἶναι ἐξαιρετικὰ ἀποτελεσματικὰ θεωρητικὰ ἐργαλεῖα.

Εἶναι συνεπῶς μεγάλη ἡ χαρά μου ποὺ μπορῶ νὰ κλείσω τὸ ἄρθρο αὐτὸ ἐπισημαίνοντας ὅτι ἡ στάση πρὸς τὰ πρότυπα καὶ τὴ συμφωνία τους μὲ τὴν πραγματικότητα ἦταν ἀκριβῶς τοῦ τύπου, ποὺ ἐθεωρεῖτο ἀναγκαῖος στὴν Ἀκαδημία τοῦ Πλάτωνος (πρβλ. Ἀριστ. *Μεταφ.* 990 a). Γιατὶ ἡ «ὑπόθεση ἐργασίας» μου στὸ ἄρθρο αὐτὸ ἦταν ἀκριβῶς ὅτι ὁ Πλάτων στὸν *Τίμαιο* χρησιμοποιεῖ τὴ θεωρία τοῦ Εὐδόξου ὡς πλαίσιο ἀναφορᾶς γιὰ τὴν ἀποσπασματική του ἀστρονομία. Στὸν *Τίμαιο* ἀνεκάλυψα τὶς νύξεις γιὰ τὰ Πυθαγορικὰ *τριπλᾶ* καὶ γιὰ τὴ λοξότητα τῆς ἐκλειπτικῆς τοῦ Εὐδόξου (καὶ ἡ *μεγάλη ἁρμονία* τοῦ Πλάτωνος δίνει δύο κλίμακες στὴν *ἀράχνη*, τὴ μουσικὴ καὶ τὴ γεωμετρικὴ ἁρμονικὴ κλίμακα γιὰ τὶς γωνιακὲς μετρήσεις). Ὁ Πλάτων ὅμως στὸν *Τίμαιο* 37 c χρησιμοποιεῖ τὸν ὅρο *ἄγαλμα* γιὰ τὴ σχέση μεταξὺ τῆς «ψυχῆς τοῦ κόσμου» καὶ τῶν πλανητῶν, καὶ ἔχω δείξει ἀλλοῦ (*Plato's Agalma of the Eternal Gods,* «Yearbook of the Philosophical Society of Finland» 1969) ὅτι ὁ ὅρος *ἄγαλμα* εἶναι βαρυσήμαντη φιλοσοφικὴ μεταφορὰ στὸν Πλάτωνα, ἐξαιρετικὰ πρόσφορη νὰ περιγράψῃ τὴ σχέση μεταξὺ τῶν «αἰωνίων» καὶ «προσκαίρων» χρονικῶν ὀντοτήτων. Τὸ «ἄγαλμα» στὰ χέρια τοῦ Πλάτωνος ἀπὸ θρησκευτικὴ ἔγινε φιλοσοφικὴ ἔννοια. Διατήρησε ὅμως μερικὲς ἀπὸ τὶς ἀρχικὲς θρησκευτικὲς ἀποχρώσεις της. Τὸ «ἄγαλμα» στὸν *Τίμαιο* 37 c εἶναι: 1) εἰκόνα τοῦ παραδείγματός του, καὶ ὁ Δημιουργὸς προσπαθεῖ νὰ τὸ κάνῃ ἀκόμα ὁμοιότερο στὸ ἀρχέτυπό του, 2) κινούμενο καὶ ἔμψυχο, 3) ἀντικείμενο χαρᾶς τοῦ Δημιουργοῦ του, καὶ 4) φαίνεται νὰ λειτουργῇ ὅπως τὰ λατρευτικὰ ἀγάλματα, στὰ ὁποῖα ὑποτίθεται ὅτι μετέχουν οἱ θεότητες τουλάχιστον σὲ στιγμὲς ἐξάρσεως τῆς λατρείας.

Στὴν πνευματικὴ αὐτὴν ἀτμόσφαιρα παρουσίασε ὁ Εὔδοξος ἕνα γεωμετρικὸ πρότυπο ποὺ «ἔσωζε τὰ φαινόμενα», ἀλλὰ σὲ ἀνταπόκριση μὲ τὴν πραγματικότητα μόνο σὲ δεδομένους χρόνους. Ὅπως τόσοι μεταγενέστεροι μαθηματικοὶ ὁ Εὔδοξος δημιούργησε ἕνα ὁρατὸ πρότυπο, ποὺ ἡ εὑρετικὴ (μεθοδολογικὴ) ἀξία του ἔγκειται ἀκριβῶς στὸ γεγονός, ὅτι ὁ,τιδήποτε γίνεται μὲ τὸ πρότυπο ἐξυπακούει ἐπὶ μέρους λύσεις σὲ προβλήματα, ποὺ ἀνακαλύπτονται στὴν πραγματικότητα. Δὲν εἶναι τὸ ἴδιο αἰώνιο ἰδανικό, στὸ ὁποῖο ἐπίσης σκόπευαν φιλόσοφοι ὅπως ὁ Leibniz καὶ ὁ Wittgenstein μὲ τὶς προσκολλημένες στὸν «σημαντικὸ» ἀτομισμὸ θεωρίες τους;

Hauho /Finland Erkka Maula