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ARISTOTLE & THE PHILOSOPHY OF MATHEMATICS*

Looking into the Aristotelian bibliography one seldom finds that which the Stagirite philosopher offered to what we nowdays call Philosophy of Mathematics. However, comparing his with Plato's contribution, where Mathematics is the very introduction to his Dialectic (entry to the Academy presupposed a deep knowledge of Mathematics), Aristotle's is not behind neither in originality nor in importance. This contribution allowes one to see another fine dimension of the universality of Aristotelian thought.

It is known Aristotle (384-322) was a student for twenty years in the Academy, where the mathematician Eudoxus from Knidos (408?-355?) replaced Plato (428/7-348/7) while he was absent in Sicely. For one to understand the Aristotelian Mathematics, one first has to look into the ideas of his teacher[1]. For Plato the beginning of mathematical thinking is the distinction between being and becoming, that is between Plato's Ideas taken as archetypes and their images, the Homoiomata. Mathematical concepts belong for Plato to the world of homoiomata, as exact images of Ideas, unlike the real (empirical) objects which are imperfect images of archetypes. Whatever mathematical thought produces is precisely the transition from becoming towards being. According to Plato the Ideas are held in the soul by recollection from the time she was in the world of truth, that is in the world of pure Ideas, before the entrance to the perishing body and that recollection is provided by Mathematics. This last science goes beyond the physical world even if does not attain the world of Ideas.

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1. The best known works which deal with the Mathematics of Plato and Aristotle are: J. L. Heiberg, *Gesch. der Math. Wiss.*, Leipzig 1904, T. L. Heath, *Mathematics in Aristotle*, Oxford 1949, H. G. Apostle, *Aristotle's Philosophy of Mathematics*, Chicago 1952, K. Reidemeister, *Das exakte Denken der Griechen*, Darmstadt, Wiss. Buchgesellschaft, 1972[2], A. Szabó, *Die Anfänge der Griechischen Mathematik*, Akademiai Kiado, München-Wien, 1969. In these works one finds also the references to the texts of Plato and Aristotle.

Aristotle, to this Platonic view of Mathematics, puts aside his own theory according to which the Forms (or Eide), in place of Ideas, are part of the real objects. Form and matter are inseparable, they cannot exist the one without the other and it is generally only thought (*dianoia*) that can make the distinction. Therefore, according to this view, Mathematics describes, e.g. in the mathematical space, properties and relations which are contained in the objects of the empirical world. Hence, these objects in the Aristotelian view have a part of matter and a part of Form.

Among Forms there are some which can and others that cannot conceptually distinguished from its corresponding material part. In mathematical objects, which result by abstraction, one overlooks matter and occupies oneself with those Forms that can be conceptually distinguished from matter. As a result we obtain, in the case of Geometry, the mathematical figures of the geometrical Space. The geometrical figures are real entities but their nature, of interest to the philosopher, does not at all concern the mathematician. In Geometry only properties and relations of figures are of interest to Mathematics. According to Aristotle, «Mathematics distinguishes itself from Physics only insofar that, although dealing with real, perceptible, objects it is exclusively concerned with a certain restricted class of properties of these objects and systematically leaves all other properties out of consideration because from its peculiar point of view they are purely accidental»[2].

After this exposition of Aristotle's views on Mathematics, we will look into some of his main achievements in the Philosophy of Mathematics.

The first is the analysis of the structure of science, the so-called Theory of Science. One will find this in the *Posterior Analytics*, a work which, as we know, has as model Mathematics. It is here worth noticing that the terms Theory and Philosophy in the Aristotelian texts are almost synonymous. Theories are true answers to questions starting with why, i. e. they are what we nowdays call theories founded on Reason.

According to the Aristotelian scholar W. D. Ross, in all periods of Aristotle's thought Mathematics was the only pure science; all other sciences, named as such by the Stagirite, «have the name of science by courtesy, since they deal with matters where contingency plays a quite signi-

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2. E. W. Beth, *Mathematical Thought*, Dordrecht-Holland, D. Reidel Publ. Co. 1965, 25.

ficant role»[3]. There is not unanimity that Euclid (330?-273?) wrote his *Elements* of Geometry and Arithmetic following the Aristotelian Theory of Science. In this last theory, which has its origin in the Dialectic of the Eleatics[4], the terms —which as mathematical objects are real entities— like their properties and relations contain a finite number of principles, so that all other terms and propositions can be defined, respectively can be logical deduced (concluded), from these principles which are valid independent of their conclusions; the mathematician comes to know them in the way that he apprehends universals, that is with intuitive induction through perception. It was this Aristotelian point of view which urged I. Kant (1724-1804) to develope his idea of pure intuition and to envisage Geometry as an a priori science. We have to notice here that we nowdays consider als principles also those, which are verified by their conclusions, i. e. those which play an explanatory role, justified by the conclusions drawn from them.

It is interesting in this place to quote the so-called Principle of Absolute, a principle which one finds implicit in the Aristotelian text. We owe the exact and general formulation of this principle to the modern mathematician E. W. Beth. This principle leads to several other principles of Aristotle as well as to principles of earlier philosophers. The application of the principle of Absolute is what the Stagirite calls induction and what his teacher Plato calls progressing to the anhypotheton. This principle one can formulate as follows: Suppose we have a relation which holds for couples from a given domain of entities; then there is an absolute entity such that the named relation holds for every couple of any given entity (different from the absolute) and the absolute one, but does not hold for couples of the absolute and any other from the given entities[5].

We then have the Aristotelian distinction between the two aspects of infinity; that of the actual and that of the potential[6]. This is illustrated in the third book of the *Physics*. What we are in fact distinguishing here is on the one hand an entity which already is infinite (ac-

3. W. D. Ross, *Aristotle's "Prior and Posterior Analytics"*, Oxford University Press, 1965[3], 14.

4. A. Szabó, *Die Philosophie der Eleaten und der Aufbau von Euklids "Elementen"*, «Φιλοσοφία» 1 (Athens 1971), 194-228.

5. E. W. Beth, *The Foundations of Mathematics*, North-Holland Publ. Co. 1968, 9.

6. W. D. Ross, *Aristotle's "Physics"*, Oxford University Press 1958[4], 111, 6-8.

tual infinity) and on the other hand a variable entity which is becoming infinite (potential infinity). The first is rejected by Aristotle because it leads to antinomies (or paradoxes), a rejection which is based on suitable argumentation. Much of this argumentation is even of present day interest. Thus, according to the Stagirite, the existence of the totality of all natural numbers isn't acceptable, though the infinite sequence of the natural numbers is acceptable.

The rejection of the actual infinity had not been disputed, except in a few isolated cases, for more than two millenia up to and nearly including the nineteenth century. The great mathematician C. F. Gauss (1777-1855), so called Prince of mathematicians, came to agree with Aristotle's view[7]. Ever since though Set Theory, created by G. Cantor (1845-1918), short time before the end of nineteenth century, with the introduction of the actual infinite becomes entangled with the antinomies which Aristotle brought into our attention. Eventually, these antinomies caused deep disagreements among mathematicians, disagreements which until now have not been fully resolved. It is worth noticing that, according to I. M. Bochenski, one of the antinomies of the new theory, the one about «the existence of a class containing all entities», is referring to what had been already known to Aristotle. The Stagirite says in the *Metaphysics*: «But there cannot be one genus of (all) things, neither the One nor Being; for on the one hand the differentiae of every genus must each both be and be one, while on the other hand it is impossible for (either any species of the genus or) the genus apart from its species to be predicated of its proper differentiae, so that if the One or Being is a genus no differentiae will be either being or one»[8]. Notwithstanding, in a recent paper, M. F. Lowe sustains that in the above passage Aristotle «is not talking about the universal class of modern Set Theory, but about a much smaller one, namely a purported universal genus»[9].

Even the same concept of potential infinite is direct connected with the problem of its e x i s t e n c e. According to Aristotle the demonstrative reason for the above existence is based on the infinite divisibility of a magnitude.

7. One reads in a letter (1831) of Gauss addressed to the astronomer Schumacher: «So protestiere ich gegen den Gebrauch einer unendlichen Grösse als einer vollendeten, welches in der Mathematik niemals erlaubt ist». Even later (1887). E. E. Kummer, on occasion of a speech about Leibniz, maintained that actual infinity «nicht Gegenstand der Mathematischen Wissenschaften sein kann».

8. *Metaphysics* 998b 22-27, translated by W. Jaeger (Oxford 1957).

9. *Aristotle on Being and the One*, AGPh 59 (1977), 44-55.

Next follows the Aristotelian concept of continuity, which makes the central point in the so-called Zeno's (490?-430?) paradoxes and which is nothing more than an aspect of the actual infinite. Using his concept of continuity Aristotle refuses Zeno's argumentation and gives a detailed analysis about the nature of Zeno's paradoxes. Basing on the intuition of the physical world Aristotle not only conceived the concept of continuity, but he aimed also to find the connective link among Spase, Time and Motion. One finds Aristotle's development concerning continuity in the fifth and sixth books of the *Physics*.

Now although R. Dedekind (1831-1916), by the application of his own mathematical concept of continuity from Geometry to the Arithmetic of Real Numbers, uses a part of the Aristotelian physical definition of continuity —that of the identity of the limits (extremities)— he does not accepts Aristotle's conception on continuity as a whole. For example, following these last conceptions, the points on a line they not constitute a continuum, nor the mentioned points are constituent parts of a line. It is noteworthy that Eudoxus, before Aristotle, proposed an other solution for the continuity paradoxes, i. e. his famous Theory of Proportions. As a matter of fact, this last theory suggested Aristotle to consider Metaphysics as a General Ontology. On the other hand that same theory, which is a deductive theory of an utmost generality, was used by Dedekind as the starting point for his foundation of the Real Numbers. However Aristotle did not follow the Eudoxian thesis, although he is continually referring in his works to Eudoxus' Theory of Proportions[10]. Here one must notice that Aristotle didn't consider Mathematics as an accurate representation of Reality, but he maintained that between Reality and Mathematics there are the Physical Sciences.

Also in Aristotle's *Prior Analytics* one must search for the origin of the contemporary development of Mathematical or Symbolic Logic, as well as that of the Logic of Modality and the so-called Definitions by Abstraction. This is grounded on the fact that both some of the concepts and forms of Symbolic Logic are already met in Aristotle's work and the methods used by him are analogous to those of contemporary Logic; also that Aristotle's Logic of Modality, as well as his Definitions by Abstraction, play an important role in Modern Logic[11].

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10. Beth, *The Foundations of Mathematics*, 55.
11. Beth, *Mathematical Thought*, 25.

In his recent outline on *Ancient Formal Logic* (1968) I. M. Bochenski, writing about Aristotle's tremendous achievement to establish first the laws of Thought —an achievement which dominated Western Philosophy for more than twenty centuries— quotes: "we owe to Aristotle fundamental ideas on which Logic is still working today".

Besides, a reproduction of Aristotle's Logic of Modality has been undertaken in our days (1963) by S. McCall who presented in his book *Aristotle's Modal Syllogisms* a formalized system which, according to his own belief, "agrees exactly with Aristotle's at all important points".

The first attempt to state this logic as a formalized deductive system we owe to J. Lukasiewicz. About this attempt S. McCall remarks in his above mentioned book: "Yet although Lukasiewicz write in order to explain the difficulties and correct the errors in Aristotle, it is impossible not to feel that a formal system which follows Aristotle's own logical insights more closely could be constructed".

On the other hand the Aristotelian Theory of Truth is the starting point for what nowdays is called «Correspondence Theory» for Truth[12]. In short, Aristotle's theory of truth is dealing on the analysis to the question «when a statement is true?». Every such statement says that something is in such or not such a state, that a state is true or false for that thing. The statement is true when what it says is the case and false when it is not. In Aristotle's own words: *ἔτι πᾶν τὸ διανοητὸν καὶ νοητὸν ἡ διάνοια ἢ κατάφησιν ἢ ἀπόφησιν - τοῦτο δὲ ἐξ ὁρισμοῦ δῆλον - ὅταν ἀληθεύῃ ἢ ψεύδηται· ὅταν μὲν ὡδὶ συνθῇ φᾶσα ἢ ἀποφᾶσα, ἀληθεύει, ὅταν δὲ ὡδί, ψεύδεται* (*Μετ.* 1012 a 3-5).

In this way we have a correspondence of a statement and something else; truth is a predicate, a relation, between a statement and something which is not a statement. We have so the separation of all statements in two categories, excluding each other, the category of the true statements and that of the false ones. It is obvious that this theory does not presupposes a one-to-one correspondence between statements and their objects, not it presupposes that a statement can or cannot express fully its object.

There is also not a criterion about truth- and falsehood, i. e. the above theory doesn't give a way to decide when a statement is true and, in case we believe that some statement is true, if it is really true.

12. S. Körner, *What is Philosophy?*, London, Allen Lane 1969. (Page 126 in the German Translation under the title *Grundfragen der Philosophie*, München, P. List, 1970).

Aristotle, in the *Metaphysics*, speaks also about the concept of b e l i e f und discusses the famous paradoxe of somebody who pretends that all his sayings are false, so contradicting to himself — the so-called Liar paradoxe of Epimenides.

To Aristotle's theory of Truth, G. W. Leibniz (1646-1716) added that truth is whatever is the case in the real or in any possible World. We have thus the c o n t i g e n t statements, i. e. those which might be true in the real world, the l o g i c a l l y true statements, which hold in all possible worlds, and the ones that are true only in some possible world, different from the real, the i d e a l t r u e ones.

The theory of Truth is nowdays studied from prominent philosophers and mathematicians, those who are mainly occupied with Mathematical Logic[13]. Among them we mention especially A. Tarski whose theory is grounded on the Aristotelian Correspondence Theory as it was completed by Leibniz. We mention here that the analysis which Tarski gives about the Liar paradoxe has a great significance not only from mathematical but also from a pure philosophical point of view.

We then have the Aristotelian Theory of Abstraction. As is already mentioned, Aristotle gets the mathematical things by a b s t r a c t i o n; notwithstanding one must not confuse abstraction with what is called g e n e r a l i z a t i o n b y i n d u c t i o n. In Aristotle's theory of abstraction one has to search for the roots of the distinction of a b s t r a c t i o n in three categories: The physical, the mathematical and the ontological. As a matter of fact, this distinction results by f u s i o n of both Aristotelian theories, those of Abstraction and of Induction.

Finally, certain ideas with which people that have been occupied and still occupy themselves on Foundations of Mathematics go back to Aristotle. For example:

a) The validity of the logical law of the excluded middle (tertium non datur) for mathematical reasoning, where the concept of i n f i n i t y plays a most significant role. This law is fundamental for i n d i r e c t p r o o f. Obviously, the use of the principle of tertium non datur for not finite domains is influenced by a generalization of a procedure which is legitimate for finite domains. On the other hand this principle has been a powerful logical motive for the common conviction that every mathematical pro-

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13. See A. Tarski's publications on the subject; also W. Stegmüller, *Das ABC der modernen Logik und Semantik*, Berlin-Heidelberg-N. York, Springer 1974.

blem has positive or negative solution. As a characteristic manifestation of this conviction one has to consider the famous lecture delivered by D. Hilbert (1862-1943) before the Second International Congress of Mathematicians at Paris (1900). Hilbert directed there the attention to some of the most renown problems for their solution and expressed his firm conviction that «every definite mathematical problem must necessarily be susceptible of an exact settlement either in the form of an actual answer, to the question asked, or by proof of the impossibility of its solution». As it is known, this conviction has been nowdays shaken by a so-called incompletability theorem which we owe to K. Gödel (1931).

At the beginning of our century L. E. J. Brouwer, the leader of the Intuitionistic School, rejected (1927) in his foundation of the intuitionistic Mathematics even the unrestricted application of Aristotle's principle of tertium non datur. His first attempts to abolish this law from mathematical reasoning are already dated since the year 1908.

b) The process of free choise sequences, i.e. free from the concept of any law for the formation of the terms for finite or infinite sequences[14]. Nowdays this process is used, by the adherents of the Intuitionistic School, especially for the definition of continuity. Brouwer maintains that a not denumerable continuum can be obtained as a «medium of free development»; that is to say, the real numbers are not ready, except those which are already defined by a law, but they develop as free choise sequences.

c) The distinction of entities in several types or categories[15], i. e. the idea that certain properties can be maeningfully predicated of some objects but not of others. To this distinction we owe the hierarchy in steps or types of all functional statements, according to the adherents of the Logistic School.

d) Aristotle's conception concerning the self-same object of Philosophy. This traditional way of philosophical consideration is, for example, followed by the modern philosopher and mathematician A. N. Whitehead in his new and original system of speculative Philosophy, as is developed by him in a series of very interesting books[16].

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14. A. A. Fraenkel and Bar-Hillel, *Foundations of Set Theory*, North-Holland Publ. Co. 1958, 249.

15. Fraenkel and Bar-Hillel, *Foundations of Set Theory*, 168.

16. Beth, *Mathematical Thought*, 66.

Ο ΑΡΙΣΤΟΤΕΛΗΣ ΚΑΙ Η ΦΙΛΟΣΟΦΙΑ ΤΩΝ ΜΑΘΗΜΑΤΙΚΩΝ

Περίληψη.

Στὴν ἐργασία αὐτή, ὕστερα ἀπὸ μία σύντομη ἔκθεση τῶν ἀπόψεων τοῦ Πλάτωνος γιὰ τὰ Μαθηματικά, ἐξετάζεται ἡ συμβολὴ τοῦ μαθητοῦ του Ἀριστοτέλους στὴν θεωρία τῆς φύσεως τῶν Μαθηματικῶν, συμβολὴ ποὺ φανερώνει μιὰ ἀκόμη ἔξοχη πτυχὴ τῆς καθολικότητας τοῦ ἀριστοτελικοῦ πνεύματος. Γιὰ τὸν Ἀριστοτέλη τὰ Μαθηματικὰ ἀσχολοῦνται μὲ τὶς ἰδιότητες καὶ σχέσεις ἐκείνων τῶν «μορφῶν» του, τὶς ὁποῖες ἠμπορεῖ κανεὶς νὰ τὶς διακρίνη ἐννοιολογικὰ ἀπὸ τὴν ἀντίστοιχη ὕλη. Ὅπως εἶναι γνωστόν, οἱ μορφὲς κατὰ τὸν Ἀριστοτέλη ἐνυπάρχουν στὴν ὕλη καὶ εἶναι ἀδιαχώριστες ἀπ' αὐτήν, μόνον δὲ ἡ διάνοια ἠμπορεῖ νὰ τὶς διακρίνη. Ἔτσι, ἂν ὁ μαθηματικὸς παραβλέπη στὰ γεωμετρικὰ σχήματα τὴν φυσική τους ὑπόσταση, ὅμως τὰ θεωρεῖ ὡς πραγματικὲς ὀντότητες, μιὰ καὶ ὡς μορφὲς περιέχονται σὲ πραγματικὰ ἀντικείμενα. Ἔργο τοῦ μαθηματικοῦ δὲν εἶναι αὐτὲς οἱ ὀντότητες καθ' ἑαυτές, ἀλλὰ οἱ ἰδιότητες καὶ οἱ σχέσεις τους.

Τὰ κύρια τώρα ἐπιτεύγματα τοῦ Ἀριστοτέλους στὴν Φιλοσοφία τῶν Μαθηματικῶν ἠμποροῦν νὰ συνοψισθοῦν στὰ ἑξῆς: Στὴν ἀρχὴ εἶναι ἡ ἀνάλυση τῆς δομῆς μιᾶς ἐπιστήμης γιὰ τὴν ὁποία ὁ Σταγιρίτης εἶχε ὡς πρότυπο τὰ Μαθηματικά. Οἱ ρίζες τῆς θεωρίας βρίσκονται στὴ Διαλεκτικὴ τῶν Ἐλεατῶν. Ἔπειτα εἶναι ἡ ἀπόρριψη τοῦ «ἐνεργείᾳ ἀπείρου», γιατὶ ἡ ἔννοια αὐτὴ ὁδηγεῖ σὲ ἀντινομίες. Ἔτσι «ἡ ὕπαρξη ὀντότητος (συνόλου) ποὺ περιέχει ὅλα τὰ ὄντα» εἶναι κατά τινας νεωτέρους γιὰ τὸν Σταγιρίτη λογικὰ ἀδύνατος. Ἀκολουθεῖ ἡ ἔννοια τοῦ «συνεχοῦς», μὲ τὴν ὁποίαν ὁ Ἀριστοτέλης ἀναιρεῖ τὴν ἐπιχειρηματολογία τοῦ Ζήνωνος στὰ παράδοξά του. Κατὰ τὴν ἔννοια αὐτὴ τὰ σημεῖα μιᾶς εὐθείας δὲν εἶναι καὶ συστατικὰ μέρη της οὔτε τὰ σημεῖα αὐτὰ ἀποτελοῦν ἕνα συνεχές. Ἄλλο θέμα εἶναι ἡ ἔννοια τῆς «ἀλήθειας», ἔννοια ποὺ δὲν ἔπαυσε καὶ σήμερον ν' ἀπασχολῆ τὴν ἔρευνα, ἰδιαίτερα στὴν Μαθηματικὴ Λογική. Τέλος, στὰ *Ἀναλυτικὰ πρότερα* βρίσκει κανεὶς τὸν πυρῆνα τῆς σύγχρονης Συμβολικῆς Λογικῆς. Στὸν Ἀριστοτέλη ἀνάγονται ἐπίσης καὶ ἰδέες ποὺ ἀπασχολοῦν τὴν σύγχρονη θεμελίωση τῶν Μαθηματικῶν. Σ' αὐτὲς καταλέγονται: ἡ πρόβαση «ἐλεύθερης ἐκλογῆς» γιὰ μιὰ ἀκολουθία, δηλ. ἐλεύθερης ἀπὸ τὴν ἔννοια ὁποιουδήποτε νόμου, πρόβαση ποὺ συναντᾶ κανεὶς στὴν Ἐνορατικὴ Σχολή. Αὐτὴ ἡ ἔννοια τοῦ ἀριστοτελικοῦ συνεχοῦς, ποὺ ἂν καὶ διαφέρει ἀπὸ τὴν ἐπικρατοῦσα σήμερα, ὡστόσο δὲν ἔπαυσε ν' ἀπο-

τελῆ τὴ βάση γιὰ νεώτερες ἔρευνες. Ἡ διάκριση ὀντοτήτων σὲ διάφορες κλάσεις καὶ τύπους, δηλ. ἡ ἰδέα ὅτι ἔχει νόημα νὰ ὁμιλοῦμε γιὰ ὡρισμένες ἰδιότητες ποὺ ἀποτελοῦν κατηγορήματα γιὰ μερικά, ἀλλὰ ὄχι γιὰ ἄλλα ἀντικείμενα — ἰδέα ποὺ ξαναβρίσκεται στὴν Λογικιστικὴ Σχολή. Ἀλλὰ καὶ αὐτὴ ἡ ἀριστοτελικὴ θεωρία τῶν «κατηγοριῶν», στὴν λογικὴ καὶ ὄχι τὴν ὀντολογική τους ἄποψη, ἀποτελεῖ στὴν ἐποχή μας θέμα ἐντατικῆς ἔρευνας.

Ἀθῆναι

Φίλων Βασιλείου
τῆς Ἀκαδημίας Ἀθηνῶν