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ARISTOTLE'S THEORY OF MATHEMATICS AS A SCIENCE OF QUANTITIES

1. History of the definition.

According to the ancient Greek thinkers, mathematics is the science which investigates the properties of quantities, and this definition was best formulated and defended by Aristotle by the use of philosophical and scientific principles. Later mathematicians accepted this definition tacitly or by habit, for philosophical aspects of mathematics were not much of a concern to them; and Gauss reaffirmed this definition in his chapter, *The Foundations of Mathematics*. From the latter part of the last century until today, however, there has been a tendency away from this definition and in the direction of what is thought to be a more general definition; for, in the opinion of recent thinkers, the notion of quantity is too limited to be applicable to much of modern mathematical research, such as projective geometry, analysis situs, theory of numbers, group theory, and the like. Bertrand Russell, in his *Introduction to Mathematical Philosophy*, pg. 195, expressed this belief as follows:

> "It used to be said that mathematics is the science of 'quantity'. 'Quantity' is a vague word, but for the sake of argument we may replace it by the word 'number'. The statement that mathematics is the science of number would be untrue in two different ways. On the one hand, there are recognized branches of mathematics which have nothing to do with number — all geometry that does not use co-ordinates or measurement, for example: projective and descriptive geometry, down to the point at which co-ordinates are introduced, does not have to do with number, or even with quantity in the sense of *greater* and *less*..."

As a consequence of such thinking there arose a number of new definitions. Briefly stated, Peirce regarded mathematics as the science which draws necessary conclusions; Russell regarded it as a set of propositions of the form "P implies Q" and identified it with logic; others gave definitions which appear to be sufficiently general but which emphasize different aspects, e.g., such as the science of order, or of certain intuitions, or of certain relations, whether of symbols or of the things themselves.

Now a fair evaluation of a definition presupposes both an understanding of the terms in that definition and knowledge of the principles according to which that definition is formulated. Unfortunately, however, criticisms of the ancient definition of mathematics fail on both counts; for the critics neither understand the terms "quantity" and "property" as these were used by the ancient Greeks, nor do they know the principles according to which the ancient definition was formulated. Contrary to the general opinion of today, the ancient definition is wide enough to include more than ninety percent of modern mathematical research. In my earlier work, *Aristotle's Philosophy of Mathematics* (Univ. of Chicago Press, 1952), I made no attempt to show that this is the case, for the aim of that work did not include this. It is the aim of this treatise to go over the ancient definition in detail, to state the principles according to which it is formulated, and to relate that definition to modern mathematical research. The main definitions given by other thinkers will be examined also. I will assume the role of a spokesman for Aristotle, supporting my statements by indicating references to his works. Knowledge of Aristotle's *Posterior Analytics*, of course, is highly desirable if one is to grasp the arguments accurately.

2. Quantity as a category; its nature, attributes, and kinds.

The key terms in the ancient definition are "science", "property", and "quantity"; so these and other allied terms should be understood. First, we shall discuss "quantity", which signifies generically the subject of mathematics.

Quantity is one of the ten categories (1b25-7), and a category as such is one of the highest genera and is therefore indefinable (1014b9-11, 1045a 36-b7). One may give a sort of formula or description of quantity for the sake of the reader or the learner (1020a 7-8), but such a formula has neither a genus nor a differentia and so is not a definition.

Of quantities, some are (*a*) essential and the others are (*b*) accidental or (*c*) indirect or (*d*) derived. First, essential quantities will be discussed.

The two immediate species of quantity are number and magnitude (4b20, 1020a8-10). A number is defined as a discrete quantity, i.e., as a quantity composed of units as ultimate elements each of which is indivisible or considered as undivided and has no position (5a23-6). Since a number as matter is a plurality of units, the least number is 2 (220a27). Since a number is immovable and is investigated as such (194a3-5, 989b32, 1077b22-30), it is actual but neither potential nor in the process of changing with respect to essence; and since that which is in the process of changing with respect to essence is not definite and hence without properties while changing (although it may have attributes if regarded as changing, as in the case of an increasing number, which remains a number but is now odd and now even), a number is finite, and as finite, it is numerable, i.e., exhaustible by counting (1020a8-9). Hence there is no greatest number; for if N is posited as the greatest number, then $N+1$, which is numerable also, would be greater than the greatest number, and this result contradicts what is posited.

How does the infinite come in? It is a process without end, as in the case of the never-ending process of dividing a line or of adding one unit after another to form a greater number. Now the number of numbers is infinite in this sense; and there are properties which belong universally to every finite group of numbers even if the number of groups is infinite in the above sense, e.g., the property $2+4+6+\ldots+2n=n(n+1)$ for every n. Then how does one demonstrate this property for every n if "every" does not signify finiteness? But just as the word "every" in "every man is mortal" does not signify a finite number of men but the nature of man is sufficient in proving man's attributes, so, in this case, if the infinite is considered as potential, the nature of a number is sufficient for the demonstration, and the principle of mathematical induction is not needed. By logic, either the equality above is true for every n, or not. If not, there is a first n, say k, for which it is not true, for n is finite. Then from $2+4+6+\ldots+2k\neq k(k+1)$ and $2+4+6+\ldots+2(k-1)=(k-1)k$ we obtain, by subtracting, $2k\neq k(k+1)-(k-1)k$, or $2k\neq 2k$, a contradiction.

If the infinite is taken as something potential, it follows that equalities such as $1+1/2+1/4+1/8+\ldots=2$ are false; for the left side is still a process and so not definite, whereas the right side is actual and hence definite, but what is potential and in process cannot be equal to what is actual. For short, let us substitute $P=Q$ for the above equality. Then one might argue as follows. If P is not equal to Q, let it be equal to $Q-R$, where R is very small. This leads to a contradiction; hence $P=Q$. Is this argument valid? The predicate "not equal" is the contradictory of "equal" and not the contrary of it; and P, if it contradicts Q, may be either actual or potential or nonexistent, for all three are included under "not equal". To assume that P must be equal to $Q-R$ if it is not equal to Q is false, for this assumption makes P something definite and actual; but it may be potential or nonexistent, and in this particular case it is potential. Hence the argument for the equality $P=Q$ is not valid.

In a similar way, the equalities $2=1.999\ldots$, $\sqrt{2}=1.41421\ldots$, and

$(1-x)^{-1}=1+x+x^2+x^3+\ldots$ for $/x/<1$ are false, although the right hand side in each case approaches the left as more terms are taken; for the notation "..." in each term on the right side signifies a process and hence a potentiality and not something which has reached an end and which is definite. Again, the fact that the results appear consistent if equalities such as P=Q in the preceding paragraph are assumed to be true is no argument that they must be true; for consistent results may follow even if some premises are false, and some results may be inconsistent with certain principles which are bypassed by a mathematician. It is true, however, that the limit of P is Q.

As for other transfinite numbers, e.g., the number of points on a line, they do not exist; for the proof of their existence depends on the existence of an actual denumerable infinity, and if such infinity does not exist, the proof rests on a false premise. Besides, certain other philosophical principles are contradicted, such as the relation of subject to attribute, as we shall see later. Anyway, the aim of this treatise is not affected even if such numbers are posited to exist, for the research which seems to depend on them is relatively small or may follow from alternative principles.

All English translations use the term "number" as a translation of the Greek term *ἀριθμός*, but nowadays the term "number" is used in a much wider sense than that of the Greek term. A better translation would be "natural number" or "cardinal number" or "positive integer" or just "integer", but an accurate translation is "a natural number greater than 1". Hereafter we shall follow the other translations and retain the term "number", but it will mean a natural number greater than 1; and there is no logical or scientific objection against this usage, although there is a departure from modern convention. An objection might arise if some things which are nowadays called "numbers" are excluded from coming under a definition of mathematics; for a definition may be inadequate, even if it is neither true nor false. Let us use the italicized term "*number*" for what is nowadays called "number", and let it apply to the so-called "real numbers", signed or unsigned, e.g., to 3.2, —6, $+\sqrt{2}$, and π.

Why did the Greeks exclude 1 as a number? According to them, mathematics presupposes certain logical and philosophical principles, and from these it follows that 1 is a principle of a number but is not a number. This conclusion does not affect the theorems which follow from the usual logical and mathematical principles, for the fact that a unit is not regarded as number does not deprive it from being added to or subtracted from or related in some other way to another unit or a number. But from the point of view of philosophy or logic there is a difference; for a number is definable and is known partly in terms of a unit, but the converse is not true, and a unit is prior in existence to a number, but no number is prior in existence to a unit. For example, the knowledge or the existence of 5 apples presupposes the knowledge or the existence of one apple, respectively, but not conversely; and, in general, the knowledge or the existence of 5 units presupposes the knowledge or the existence of one of its units, respectively, whatever that unit may be, but not conversely. If this difference is not granted, falsities may result; for 6 may not be greater than 5, e.g., 6 feet may or may not be greater than 5 yards, depending on the meaning of "greater". As one length, the length of 6 feet is less that that of 5 yards; but as a number of lengths, 6 foot-lengths is greater than 5 yard-lengths, for the units in the two numbers are regarded as lengths, without reference to the fact that a yard-length is greater than a foot-length. Further, how can 5 men be equal to 5 points, and what results when 3 points are substracted from 5 men?

A number is distinct from plurality. A number has both matter and form, and its units are its matter; and that matter is plurality without a form. Further, a number has unity, and this is caused by its form; but in plurality there is no unity. Oddness and squareness are traits of 9 because of the form of 9, for without a form there would be no one thing to which oddness and squareness would belong; and though the kinds of units as matter may differ from one instance of 9 to another, oddness and squareness would still belong to those instances of 9, evidently because of their form. Further, if two numbers are to be added to or subtracted from each other, not only must each of them have unity, but also their units must be of the same kind; for if the result is to be a number, all of the units in it must be of the same kind.

It may be thought that 9 circles are equal to 9 men because there is no difference in counting, whether the things counted are circles or men. But counting is an attribute of a man and not of a number, and as such it cannot be the cause of the form of a number and so of any trait of a number. The number 6 is even because of its form, not because one counts or adds 1 to 5 or performs any other operation. Besides, counting is a species of change, and, as it will be shown later, mathematical objects are immovable and not suject to change. Moreover, all terms in mathematical demonstrations must be mathematical (71b19-24, 74b5-12, 75a38-b6); but "counting", like "thinking about numbers", although directed to mathematical objects, is not a mathematical term.

But there is one-to-one correspondence between 9 circles and 9 men, and between 9 men and 9 colors; and it would appear that similarity may be used to define sameness of number. According to Russell, we first define the number of a class in terms of similarity, then we define a number in terms of the number of some class; and since the number of a class is the class of all those classes that are similar to that class, 9 as a number turns out to be a certain class that is the class of all classes which are similar to that class. Apart from the artificiality of this definition, one cannot see how such a definition can be applied at all to physics, chemistry, engineering, and other fields. If it is inapplicable, who should be concerned with numbers which are applicable? Moreover, there are certain logical and metaphysical difficulties with the above definition and the concept of a class.

The other species of essential quantity is magnitude; and essential magnitudes are further divided into lines, surfaces, solids. Lines are one-dimensional magnitudes, and they may be straight or curved or mixtures of the two. Surfaces are two-dimensional magnitudes, and they may be plane or curved or mixtures of the two. Solids are three-dimensional magnitudes, but their differentiae are not like those of lines and surfaces; for the differentiae of solids are their limits only, which are surfaces, whereas the straightness or curvature of a line is a trait of the line as one-dimensional and not of the line's limits, which are points, and the planeness or curvature of a surface is a trait of the surface as two-dimensional and not of the surface's limits, which are lines. Density is not a differentia of a solid but of a physical body, which is not a mathematical object. Derived magnitudes will be treated later.

Lines have lengths, surfaces have areas, and solids have volumes. Each of the terms "length", "area", and "volume" may be used in two senses. In one sense, the length of a line is what one might call the "extent" or "amount" or "quantity" of that line, but without reference to measurement by any unit; and, in this sense, A as a length is related to B as a length by being either equal to or greater than or less than B. If, however, A is said to be, say, double of B, then B is taken as a measure or unit, and then A is spoken of as having two units each of which is the same in length as the length of B; and if A is said to be, say, 5 feet long, then A is spoken of as having a number of units each of which is one foot long. To avoid confusion, let the italicized term "*length*" be used for length as a *number*, which presupposes a length as a unit, e.g., a foot or a mile. The interval between points A and B is defined as the length of the straight line connecting A and B; and the distance between A and B is defined as the *length* of the straight line connecting A and B. In a similar way, the term "area" will signify the extent or amount of a surface without reference to any unit, but the term "*area*" will signify a *number* of unit areas, such as 5 or π square feet; and the term "volume" will signify the extent or amount of a solid without reference to any unit, but the term "*volume*" will signify a *number* of unit volumes, such as 5 or 7.2 cubic feet. Expressions such as "A is 3.12 times B" will be discussed later.

For the modern expression "zero length" Aristotle would use "no length" or "privation of length"; and let this be the convention here. The same applies to "zero volume", "zero area", and the rest.

It is evident from what has been said that knowledge of a *length* or an *area* or a *volume* presupposes knowledge of the corresponding unit in each case. If one does not know the length of a yard, he cannot know the *length* of 5 yards or 5 yards as a *length*. Further, the same length may be expressed as *length* in various ways, e.g., as "2 yards" or as "6 feet" or as "72 inches"; and although it is the same length that is *numbered*, the three *numbers* differ as *numbers*, for what is taken as a unit in these *numbers* is not absolute but relative to choice. Further, the unit of a *length*, although it is taken as indivisible or as undivided as a unit, is divisible as a nature, which is a length; for a length is a magnitude and hence a quantity, and a quantity is divisible (1020a7-8). Similar remarks apply to areas and to volumes when these are expressed as *areas* and *volumes*, respectively, and also to their corresponding units. Again, any two lines, regardless of their forms, are comparable with respect to their lengths (i.e., they are either equal or unequal to each other), otherwise it would be impossible to truly say that a given straight line is either equal or unequal to the circumference of a given circle; and similar remarks apply to surface and to volumes.

Is a line made out of points? If it were, it would be reducible to points as its matter just as a number is reducible to its units as its matter; and, similarly, a surface would be reducible to lines and ultimately to points, and a solid would be reducible to surfaces and ultimately to points. So since lines to points as surfaces are to lines, and since surfaces are to lines as solids are to surfaces, an argument for or against reducibility in one of the three are cases is, respectively, an argument for or against the other two cases.

Since a material part of a thing is saparable in existence from that thing but an attribute or a trait is inseparable in existence from it (except for the intellect in the case of a man, 413b24-7), one may inquire whether points are so separable from a line in which they exist. In the case of a solid or a body, which has a surface, if the surface were so separable from the solid or the body, there would exist a solid without a surface and a body without a surface. Now one may abstract (i.e., separate in thought or consider only one trait of a thing and not the whole thing) a surface from a body, for this is necessary in investigating mathematical properties of surfaces qua surfaces without reference to the things in which they exist, but in existence one cannot separate the surface from the body; so bodies without surfaces cannot exist. Since in existence a surface cannot be separated from the body in which it exists, it cannot be a material part of that body. Further, a surface is an attribute of a body, and it is the form or part of the form of a solid; and neither an attribute of a thing nor a form or a part of it can be a material part of that thing. Moreover, since a surface as a part of a body has no volume and hence no weight, the body itself, if made out of surfaces, would have no weight, for no sum of surfaces each with no weight can yield something with weight. Further, there would exist no congruent bodies which differ in weight; for congruent bodies may be regarded as composed of the same congruent surfaces, and these surfaces, having no weight, would not cause a difference in the weights of the bodies. But congruent bodies with different weights exist. Again, if one were to assume that surfaces have weight, then lines and ultimately points would have weight; for, as parts, lines are to surfaces as surfaces are to bodies, and points are to lines as lines are to surfaces. If so, a point, too, would be divisible, for it would have weight, which is divisible.

These and many other arguments may be brought against the reducibility of a body or a solid to surfaces as material parts. So since a solid cannot be so reduced to surfaces, in a similar way a surface cannot be so reduced to lines and a line cannot be so reduced to points. In the case of a line, the following two arguments may be added.

(1) Let an open line segment S (i.e., a line segment without end-points) be taken, say, the segment from $+1$ to $+2$ on the x-axis. Those thinkers who posit a line segment as consisting of points and of *nothing else* and regard any two points in S as ordered (i.e., one of the two points as being to the right of the other) arrive at the conclusion that there is no first point on S at the left side of S, in fact, nothing first which begins the line segment. But if an open line segment consists *only* of points and if *every* point has position relative to every other point, it is impossible that there should be nothing first at the left or at the right of that line. Further, if a point P at the origin starts moving to the right with uniform speed, it will be to the left of S at some moment M_1 but within S at a later moment M_2; but these thinkers, positing time as consisting of moments and of nothing else, will have to conclude that P will never (i.e., at no moment) cross S. This conslusion, too, is false and beyond imagination. So since a line does not consist of points only, and since a point cannot exist apart from a line, it follows that a line is a principle and also an element which is divisible into its own kind (1014a 30-1) and is not reducible to points. Thus points are limits or divisions of lines (1002a18-20, 1060b19) and not material parts of a line.

(2) Let us assume a one-to-one correspondence between the points on a straight line one foot long and the *numbers* from 0 to 1. The infinite decimal .1414... may be replaced by the equal series $.1+.04+.001+.0004+\ldots$ Now

any term of this series presupposes the continuum, for it is a line and not a point; e.g., .04 in the case of a line is a part of one foot and hence is a line, which is continuous, and not a point or a limit. Consequently, any point on that foot-line, if stated as a decimal in the above manner, presupposes the existence of a line as a part and so of the principle of that line, i.e., of a line of length one foot, for the part is known in terms of the whole (1034b-1035b14); and, similarly, the number .04 as something corresponding to a point presupposes the interval from 0 to .4, and that interval is a continuum and presupposes the continuum from 0 to 1, for the interval from 0 to .04 is a part of 1 as an interval and is known as four-hundredths of 1, and 1 is a principle, like the foot-line. It follows, then, that a unit line or any unit magnitude or interval cannot be reduced to points or to *numbers* but must be posited as a principle.

Now just as a surface in a solid is the form or part of the form of that solid, so is a line related to a surface and a point to a line. The form of a thing cannot be the thing's matter, and conversely. A sphere is defined as a solid bounded by a certain kind of surface, and a triangle is similarly defined in terms of straight lines; so the spherical surface is a part of the form of the sphere and not a material part, and the three straight lines in a triangle are similarly related to the triangle.

Quantities of the same kind have the property of being either equal or unequal to each other (6a26-35). For example, if A and B are lines, A is either equal or unequal to B, and if unequal to B, it is either greater or less than B; and if A and B are numbers whose units are of the same kind, they are similarly related. The qualification "of the same kind" is necessary, for a line is neither equal nor unequal to an angle or a surface or a solid or a number, and, in general, quantities which are not of the same kind are neither equal nor unequal to each other. In some cases, however, it is possible for two incomparable quantities to become comparable, provided that they or their units are expressed more generically, or else, differently. For example, 5 oranges are not equal to 5 apples, but if the unit in each case is stated as "a fruit", then the two numbers are equal, for each of those numbers is now 5 fruits. In this sense, 5 lines are equal to 5 solids, if the unit in each number is stated as "a magnitude", for both lines and solids are magnitudes, at least insofar as both are infinitely divisible. But one may say that the term "magnitude" is analogously predicable of a line and of a solid, as in the case of the term "being" when predicable of things under different categories. The problem appears somewhat difficult if the two numbers are 5 fathers and 5 colors, for the term "being" or "thing", which is predicable of a father and of a color, is not univocally predicable; and in that case one may wonder whether the two numbers are comparable. The problem appears even more difficult if one were to consider a line and a color and a relation, when taken together, as being an instance of the number 3; for these three things as units have nothing essential in common. Perhaps one would be least inclined to say that they form a number, unless one uses the expression "3 genera" or something of this sort (1088a10-4).

Since not any two quantities are either equal or unequal to each other, the terms "equal", "unequal", "greater", and "less" are analogous predicates in statements which concern all quantities (76a37-b2, 1061b18-25). The same applies to "ratio", "proportion", and the like. For example, the statement "If A is greater than B and B is greater than C, then A is greater than C" must be taken in a disjunctive manner, for A, B, and C are not just quantities, but all of them are either lines, or angles, of surfaces, or men, or colors, etc.; and no two things which are not both quantities of the same kind can be truly called "equal" or "unequal" unless the term "equal" or "unequal" has a different meaning. But any two things whatever are either equal or not equal, and they are either unequal or not unequal; for "equal" and "not equal" are contradictory predicates and are therefore subject to the principle of contradiction, whereas "equal" and "unequal" are contrary predicates and are predicable only of comparable quantities, which are within a genus (1055b8-11). Thus inequality is an attribute of a quantity only, whereas non-equality may be an attribute of a quantity and is always an attribute of a thing which is not a quantity. For example, a line is not equal to a number or to a man, and a man is not equal to a thing, whatever that thing may be.

Since only physical bodies are movable (for only these have in themselves physical matter, which is the principle of being movable, 1025b18-21, 1037a 14-16, 1059b16-18), quantity, not having such a principle, is immovable (193b22-35, 698a26, 989b32). Hence counting and dividing and performing other operations, such as moving one of two triangles in a certain manner to prove congruence and rotating axes and the like, although linguistically convenient or psychologically useful for the student, are neither causes of quantities or of their attributes nor necessary in mathematical demonstrations. The statement "If equals are subtracted from equals, the remainders are equal" may be replaced by "If parts p and q of equal quantities P and Q are equal, the remaining parts (P—p) and (Q—q) are equal", and the same applies to the associative principle $(a+b)+c=a+(b+c)$ and to other statements in which motions or operations in general are mentioned. One may prove a truth by using operations, but he does not demonstrate it, i.e., he does not prove it through the cause and without irrelevant material; for a demonstration presupposes principles of philosophy and of science, but a proof does not necessarily do so. The fact that we arrive at the same theorems, whether we use a demonstration or a non-demonstrative proof, does not eliminate the difference between a proof and a demonstration; for the aim of a science is scientific knowledge, which is knowledge through the cause alone and is therefore demonstrative knowledge, whereas the result of a non-demonstrative proof cannot be such knowledge.

As for mathematical physics, which Aristotle calls "mechanics", it is concerned with physical bodies qua movable and qua quantitative; for since physical bodies have quantities, quantities and mathematical attributes are applicable to physical bodies, so these may be investigated mathematically (847a24-8). Since physical bodies are subject to change, their quantities too are subject to change, but indirectly or accidentally. For that which is changed essentially can be only a physical body; and if that change is with respect to quantity, the change in quantity belongs to a physical body qua physical and hence is caused not by the changeability of quantity itself but by the changeability of the physical body with respect to quantity (226a23-32). Thus a change in quantity is an attribute of a physical body only and not of quantity qua quantity, but one may truly say that a quantity is changed indirectly or that change is an indirect attribute of quantity, for the quantity is changed not because of itself but because of something else in which it exists as an attribute.

But is it not true that quantities themselves change sometimes, e.g., when 2 quarts of water are added to 2 quarts of alcohol? For the sake of argument, let the mixture be 3 quarts. This fact, too, comes under mathematical physics and not under mathematics. The changeability of quantity is caused by the changeability of the corresponding units qua physical natures when mixed, i.e., qua alcohol and qua water when mixed, so the quantity of the mixture is changed indirectly. Further, quantitative investigation of such changes presupposes immovable quantities. For example, to find the volume resulting when 5 quarts of water are added to 5 quarts of alcohol, we proceed as follows:

$$\frac{5+5}{2+2} = \frac{x}{3}, \qquad (1)$$

hence

$$7\frac{1}{2} = x \qquad (2)$$

Step (2) follows from step (1) by multiplying two equal sides by 3, i.e., by adding equals to themselves twice and assuming the results to be equal. Thus the units in such addition are not subject to change, and neither is the principle that sums of equals are equal. In general, scientific knowledge of things in motion presupposes ultimately knowledge of the principles of motion, and those principles are immovable. To *know* that a body moves from A to B, one must *know* what motion is and that motion itself is immovable, otherwise it would be changing to its contrary, which is rest; and A and B must be *known* in a similar way.

The species of essential quantity have been considered. But there are other kinds of quantity also, and these depend on essential quantity (5a38-b10, 1020a26-32). If A and B are white bodies, we may say "this white is greater than that white", for both A and B are white; but what we mean is that the volume of A is greater than that of B. Both whiteness and volume belong as attributes to A and to B, but we use the term "white" instead of the term "volume". Such usage of the term is accidental or indirect, and the white is said to be a quantity or to be greater than B accidentally or indirectly; for A would be greater than B even if it were red or were to change to red, assuming that it remained the same in other respects.

In a somewhat similar way, time and motion are said to be quantities, but indirectly. If A and B are bodies which have moved along a straight path, they may differ in volume, or in density, or in the distance they have travelled; and any one or any combination of these three as principles (volume, density, distance) may be used in measuring the motions of A and B and then comparing them. For example, if only the distance is taken as a principle of measurement and the distance travelled by A is greater than that of B, then A's motion will be said to be greater than that of B; and if all three principles are used and A's weight, which combines volume and density, is four times that of B but travels one-third of the distance travelled by B, then its motion will be said to be greater than that of B. Similar remarks apply to time. Now motion and time and density are traits of physical objects and come under physics; and when volumes and distances and measurements are attributed to them, such attribution relates mathematical attributes to physical subjects and comes under mathematical physics. But since the subjects are physical and not mathematical, such mathematical terms as "multiplication" and "being greater", which are essentially predicable of mathematical subjects only, are predicable of physical subjects indirectly and not univocally. One man is taller than another not qua man — for as substances both have the same nature — but qua having a certain quantity which is greater than that of the second man. Time and motion, however, are closer to quantities than colors or positions or affections.

One might wish to apply quantitative predicates even to quantitative attributes which are not quantities, but this would be an error. The curvature of a line, for example, is a quality and not a quantity, but it is a trait of a quantity; and as a quality it cannot be equal or unequal to the curvature of some other line. Now just as equality and inequality are proper to quantities of the same kind, so likeness and unlikeness are proper to qualities of the same kind; and corresponding to the terms "greater" and "less", which are applicable to quantities only, there are the terms "more" and "less" (using "less" in a different sense here), which are applicable to qualities but not to quantities. Accordingly, one line may be more curved than another, but its curvature as a quality is never equal or unequal to that of the other.

The numerical term "curvature", as used by modern mathematicians, means something else, namely, a certain ratio of two quantities or a certain quantity corresponding to that ratio. It is defined as the limit of $\Delta\theta / \Delta s$ as Δs approaches zero; and, in the particular case of a circle having a radius of *length* R, it is the ratio of 1 over R, or the quantity corresponding to that ratio. Let us use the italicized term "*curvature*" for this kind of curvature. Evidently, there is a one-to-one correspondence between curvature and *curvature;* but correspondence is a relation between two different things and not an identity of those things. Further, curvature is prior in definition and hence in knowledge to *curvature*, otherwise what is *curvature* the *curvature* or the measure of? Historically, mathematicians had curvature in mind when they introduced *curvature* as a ratio or as a quantity for it, and they did this for the sake of certain investigations; but if one wishes to dismiss curvature from geometry because it is known by intuition, he will have to dismiss all geometry also, in fact, all mathematics, for no scientific knowledge is possible without concepts as principles, and having concepts is impossible without intuition, as already shown in *Posterior Analytics*, B-19. If one has no concept of a circle, how can he know any of its attributes as being attributes of a circle?

There are other mathematical terms like "*curvature*", for there are many ways in which there may be a one-to-one correspondence between certain mathematical objects and their corresponding ratios or quantities. Such are "radian", "trigonometric function", "derivative", "mean ordinate", "root of an equation", "radius of convergence", "solid angle", "vector", and the like, and all of them have only mathematical terms in their definitions and hence belong to mathematics. Such terms and what they signify may be called "derived"; and they differ from terms which signify quantities indirectly, e.g., from terms like "specific gravity", "velocity", "acceleration", "work", "pressure", "moment of inertia" and the like; for all of the latter include in their definitions terms from physics and are, as such, not mathematical but composites of mathematical and physical terms and are therefore terms signifying quantities indirectly — they belong to mathematical physics.

Greek mathematicians were not much concerned with derived quantities as their subject; their main concern was with essential quantities. But derived quantities were included and were implicit in Aristotle's philosophy of mathematics. Angles and ratios were investigated and mentioned as quantities, and the kinds of units discussed by Aristotle allows investigation of derived quantities. Thus division by a unit whose nature is such as to be infinitely divisible, e.g., by a length of one foot or a unit surface or a unit of weight, was discussed by Aristotle, and since a remainder left by such division is related to the unit as a ratio of a part to a whole, such objects as 2.4 and .01 and the like were included, regardless of the lengthy terminology used. Further, Aristotle's concern in investigations was only with attributes which are properties, for only these indicate the cause, whether univocally or analogically (74a17-25, 99a15-6); so as a matter of principle he would insist on modern attempts at generalizations, as long as their formulation does not contradict philosophical principles.

The kinds of quantities have been stated. Mathematics — or pure mathematics, if you wish — has as its subject essential and derived quantities and whatever belongs to these, and it investigates properties.

Since quantity is a being, it is subject to the principles of being. Now there are principles of being qua being and also principles of being qua *known*. Philosophy is the science of being qua being, and logic is the science of a certain being qua *known*. Consequently, mathematics presupposes two kinds of principles, those of being and those of being *known*, and it is thus subordinated to philosophy and to logic. Logic, for Aristotle, includes the *Prior Analytics* and the *Posterior Analytics*, the latter being concerned with demonstration, i.e., with the nature of scientific knowledge.

Arithmetic is prior in knowledge to geometry; for the subject of arithmetic, which is number, is more abstract than that of geometry, which is magnitude. Evidently, numbers are studied without any reference to magnitudes, but magnitudes often use numbers; for we speak of 4 magnitudes in a proportion and we define a pentagon as a plane figure with 5 sides, but no mention of a magnitude is made in the study of numbers. A unit, too, which is a principle in numbers, has no position or is considered without reference to position, but a point, which is a principle in magnitudes, has position (87a31-37); hence units, if considered without reference to position, are universally applicable to points, but points are not universally applicable to units.

We may now turn to the principles according to which Aristotle formulates the definition of mathematics.

3. Criticism of recent definitions of mathematics.

The Greek term for the word "philosophy", if literally taken, means the love of wisdom, and wisdom is universal truth or knowledge of those things which great thinkers value most. Hence rational activity, which is proper to man, should have as its aim such truth as far as possible; for the dignity and value of such activity is measured by the extent to which man fulfills that aim. It is not without cause, then, that throughout the ages great men regarded scientific activity as the pursuit of such truth. Accordingly, if mathematical activity is to be worthy as scientific activity, its aim should be universal truth of the objects of mathematics in a scientific manner; and if a definition of mathematics leaves out, partially or totally, such truth as the aim of mathematical activity, to that extent it excludes what has been regarded as the worth of that activity. First, a brief examination of some recent definitions of mathematics will be undertaken. These definitions will be discussed with respect to what they state, not with respect to what their authors do after they posit them.

According to the Logistic School, mathematics is the class of all propositions of the form "p implies q", where p and q are propositions containing one or more variables, the same in the two propositions, and neither p nor q contains any constants except logical constants, e.g., implication, relation of membership, such that, relation, and truth. This definition is in accord with the belief of most present-day mathematicians who assert that mathematics is concerned not with the truth of its assumptions but with what follows from a consistent set of assumptions laid down by the mathematician.

First, the above definition excludes an important aspect of what mathematicians, prior to this century, generally regarded as mathematical knowledge, namely, the truth of the assumptions of mathematics, for these assumptions are elementary propositions and not of the form "p implies q".

Second, the definition hardly differs from the traditional definition of logic, regardless of whether modern logic is or is not wider than or superior to traditional logic, and one gains nothing by changing symbols without introducing new meanings. Further, the definition would include under mathematics, in some way or other, everything which uses reasoning or proof, and so biology, sociology, chemistry, and all disciplines which use reasoning would come under mathematics, whether as branches or as applications or in some other way. A father, too, who is trying to reason with his son would be using or applying mathematics, and so would a lawyer who is questioning a witness. Now a scientist who is not a mathematician would not object to the statement that the above disciplines and the father and the lawyer are using reasoning or logic, but he would hardly be inclined to accept the statement that his science comes under mathematics. A sociologist, for example, can likewise turn around and say that mathematics comes under sociology, for a mathematician is a product of society, he is influenced by it, he depends on it, and he cannot function without it.

Third, the unity of a science lies in the fact that the objects of that science come under one genus or are related to one aim (1003b11-15, 1094a5-9). For example, biology is concerned with living things, botany with plants, dynamics with bodies in motion due to forces, medicine with health, strategy with victory, and ethics with happiness. Now those who put forward the above definition claim that the objects with which mathematicians have been dealing come under their definition, but when we examine the objects as indicated by that definition and those with which mathematicians have been dealing we notice a wide difference. The unity of the objects as indicated by that definition is a certain implication with propositions as elements, whereas the objects as elements dealt with by mathematicians are lines, surfaces, numbers (in the modern sense), functions and the like, none of which is a proposition; and it makes no difference which of the various definitions of a proposition one chooses to adopt.

If propositions are regarded as sentences (or judgments or statements, for it makes no difference in this case) as these are defined in the dictionary, they have meaning, for they signify not themselves but usually objects which are not propositions. Lines and numbers and functions, on the other hand, have no meaning but are just things, and, as we shall see later, propositions about them do not come under the definition of mathematics as the class of propositions of the form "*p* implies *q*". If, as some say, a proposition is regarded as a class of sentences having the same meaning, how can a point or a parabola or a sphere or even the statement "Vertical angles are equal", which is just one thing, be what they call "a class" of sentences? A class is a plurality of things. If, as others say, a proposition is regarded as that which is true or false and is not a sentence but the meaning corresponding to a sentence, how does such a proposition exist? If one says that a proposition is anything which is believed or disbelieved or is supposed, he does not specify the nature or genus of a proposition. God, too, is believed or disbelieved, but he is not a proposition. Perhaps it is a fact or a nonfact (i.e., the opposite of a fact), or a Platonic Idea, or else a relation between a sentence and a fact or a nonfact or an Idea; for there seem to be no other alternatives. It cannot be an Idea; for Plato assigns existence to Ideas, and there can be no Idea corresponding to a false proposition. If it is a fact or a nonfact as such, then "true proposition" and "false proposition" would be redundant terms, for they would be synonyms of "fact" and "nonfact", respectively. Further, what would the common word "proposition" in those terms mean? A fact and the opposite nonfact have nothing in common to correspond to "proposition";

and, we may add, we cannot discourse adequately without the terms "truth" and "falsity", which signify thoughts or expressions of some kind and what these signify. It would appear, then, that a proposition is a relation, and that the two related parts, which must be understood together as a pair (8a35-b15), are (*a*) a sentence as asserting a fact or a nonfact and (*b*) the corresponding fact or nonfact as being asserted. This view makes possible the distinction between truth and falsity; for the same fact (or nonfact) may be either asserted or denied by one of two contradictory sentences, and the same sentence may assert (or deny) either a fact or the opposite of that fact. If so, then all the objects of mathematics would be relations or implications between relations. But this is actually false; for a sphere and a number and a point are not relations.

In general, regardless of the definition one may choose to adopt, since corresponding to a true elementary proposition there is a false one, and since every form "p implies q" has elementary propositions as elements, half of such elements as objects of mathematics would signify or be nonbeings; for p in the above form is not limited to what is true. But have mathematicians been concerned with nonbeings as their objects? And, we may add, propositions in any of the above senses are not limited to mathematical objects as conceived by mathematicians but are used in all sciences and are applicable to all objects.

Fourth, the analysis of p and q in "p implies q" is made in such a way as to appear to lead to mathematical objects and not to those of physics or of ethics or of any other science. But this restriction presupposes that the mathematical objects are prior to all other objects, and we are back to positions of the Pythagoreans and the Platonists, who regarded the objects of mathematics as prior to all others; and the mistakes of history repeat themselves (985b23-988a17, 1076a8-1087a25). The fact that mathematics is more accurate than physics or ethics does not imply that mathematical objects are prior to those of physics or of ethics in all respects; for, besides priority in accuracy, there is priority in universality, in value, in causation, in existence, and so on (14a26-b23, 1018b9-1019a14). Mathematical objects are attributes of physical objects, and as attributes they are not prior in existence to physical objects; and, similarly, they are not prior in value to the objects of ethics or of psychology (402a1-7). So it is a mistake to regard mathematics as the super-science or as prior to all other sciences.

Fifth, mathematical propositions are simple or elementary and not of the form "p implies q". For example, the propositions "Vertical angles are equal", "The prime natural numbers are infinite", "The derivative of e^x with respect to x is e^x" and the like are not of the form "p implies q". Are these to be excluded from mathematics? Even those which appear to have the form "*p* implies *q*" are actually elementary propositions. The theorem "If a triangle is equiangular, then it is equilateral" can be stated as "An equiangular triangle is equilateral", and the theorem "If an infinite series is convergent, then the new series obtained by multiplying each of its terms by a fixed number *k* is convergent" can be stated as "An infinite series each of whose terms is *k* times the corresponding terms of a convergent series is convergent". All such theorems are of the general form "A mathematical object with such qualifications has such a property". Thus mathematicians do not have to express any of their principles or theorems in the form "*p* implies *q*", although they may use that form in demonstrating elementary propositions; for, as a science, mathematics uses logical principles, and some of these have the form "*p* implies *q*", as in the case of demonstrating a theorem by positing its contradictory and then proving a contradiction.

Perhaps the form "*p* implies *q*" should be so taken that *p* are the axioms and *q* the theorems. If so, the objections in the preceding paragraph would seem to disappear. But this position, too, leads to difficulties. Let *r* be one of the theorems. If the set *p* is false (and let this be the case if some or all of the axioms of *p* are false), then it implies both *r* and its contradictory; for a theorem in the *Principia Mathematica* states that a false proposition implies any proposition. But would a mathematician allow a false *p* if that *p* implies that the prime natural numbers are both finite and infinite, or that vertical angles are both equal and unequal, or that continuous functions are both integrable and nonintegrable? Evidently, a false *p* must be rejected; and, since *p* must be either true or false, only a true *p* must be allowed, that is, only true axioms must be posited.

Sixth, the objects of mathematics as conceived by mathematicians cannot all be defined in terms of propositions and the so-called "logical constants" alone; in fact, many mathematical definitions as given by mathematicians do not need the term "proposition" or the use of logical constants. For example, a triangle is defined as a three-sided plane figure. Again, the number 1 as usually conceived is one definite thing and not a certain class of classes, and the same applies to $\sqrt{2}$, for, if not, 1 apple would be a certain class of classes, which is ridiculous, and the diagonal of a square whose side is one unit would be not a definite thing but a class of numbers *n* with $n^2 < 2$, and so applicability of numbers as we usually understand them and apply them to chemistry, physics, engineering and other fields would necessitate a mathematical science distinct from the one which these thinkers are positing.

In general, if the definitions of mathematical objects as given by the Logistic School are intended to be either more general than or distinct from those as given by the mathematicians (for they are neither the same nor intended to be less general), difficulties will follow. If they are distinct, they are not of the objects with which the mathematicians are concerned; and if they are more general, then most of mathematical research will have to be excluded as part of mathematics, for it is impossible to define a species in terms of the genus alone, and, in general, it is impossible to define the less general in terms of the more general alone (998b2-3, 1057b22-3). A man is not only an animal, but also rational; and rationality is not definable in terms of animal alone.

Seventh, the mathematical kind of implication, which is used by mathematicians, differs from material implication, which is used in the *Principia Mathematica*. According to mathematical implication as practiced by mathematicians, the theorems which follow from true or false mathematical axioms can have only mathematical terms. This is not the case in the *Principia Mathematica*; for the theorem which states that a false proposition implies any proposition allows any proposition with nonmathematical terms to follow from a false axiom with mathematical terms.

Difficulties arise even if the mathematical kind of implication is used by the Logistic School. If the mathematician is not interested in the truth or falsity of a set of axioms, he should have no objection if every consistent set of axioms is a part of mathematics. But would a mathematician accept a consistent set of axioms — and it is easy to produce such a set — which implies that the prime natural numbers are finite, or that a continuous function is not integrable, or that vertical angles are not equal? To regard T as a mathematical theorem in mathematics one must deny the contradictory of T, otherwise he would be violating the principle of contradiction. In general, if all consistent sets of axioms are allowed as parts of mathematics, all research will be useless; for, corresponding to any consistent set of axioms from which a theorem T follows there will be a consistent set of axioms from which the falsity of T follows.

The so-called "non-Euclidean geometries" would seem to be an exception, but the truth about these geometries is not stated clearly. It is claimed that, on the same plane, if a point lies outside a straight line (usually called "line" nowadays), the number of straight lines which can be drawn through that point and be parallel to that line may be one, or infinite, or none at all, depending on the axiom set taken, and that each of those axiom sets is consistent. This claim may be briefly examined.

Let E, L, and R be a Euclidean, a Lobachevskian, and a Riemannian straight line, respectively, and S be a straight line in the affine (or projective) geometry. Also, let each of S_1, S_∞, and S_0 be the attribute that the number of straight lines which can be drawn throught a point outside a straight line are one, infinite, and zero (or none), respectively. Now S is a genus with E, L, and R as its species; hence every trait of S belongs to E, to L, and to R. Does S_1 belong to every straight line? If "straight line" means S, this question is like the question "Does evenness belong to every natural number?", and the answer is in the negative. Evenness belongs to some but not to all natural numbers; it belongs to a species of natural numbers. Similarly, S_1 belongs to a species of S, i.e., to the E's, but not to every S, and so "S_1 belongs to S" is false. It is evident, then, that S_1 or S_∞ or S_0 does not belong to S, and that S_1 and S_∞ and S_0, which contradict each other, do not all belong to S, or to E, or to L, or to R. Accordingly, the principle of contradiction is not violated.

Now since E and L, which differ in species, are not identical but differ, they are falsely conceived if they are conceived as being identical. Consequently, they are falsely drawn — as they are often drawn in texts — if they are drawn in an identical manner; for to conceive or to draw E and L in an identical manner is like conceiving or drawing a horse and a man, which are animals but differ in species, in an identical manner. When asked to imagine or to draw S, however, one will have to imagine or draw E, or L, or R; for S is a genus, and a genus, not existing apart from its species (999a6-11, 1038a5-9, 1053b21-4), cannot be imagined or drawn *as such*, i.e., without a differentia. So if one imagines or draws E, he is not mistaken, provided he knows that he is imagining or drawing a species of S and that E is a species of S. Thus, in some sense, the concept of S is to the images of E, L, and R as a genus is to its species, or as a universal is to the particulars under it, and one cannot conceive S without an image of E or of L or of R (431b12-2a14). We may add, mathematicians often confirm the consistency of non-Euclidean geometries by using a Euclidean model. For a Riemannian plane geometry they use, for example, such a model as the surface of a sphere, thus using the term "plane" equivocally and in a sense different from that of Euclid. There is nothing wrong is using the same term in many senses, as long as one is aware of the equivocation and of the fact that the things signified in the various cases have differences and can be studied in Euclidean geometry. Actually, of course, the term "plane" in affine geometry is generic and is not used in the Euclidean sense, which is the sense in which we usually understand it. As for those who regard E, L, and R as symbols without meaning but subject only to a set of axioms, they are subject to the objections against the Formalist School, which will be considered later. I have dealt with non-Euclidean geometries in greater detail elsewhere.

Eight, it is claimed that the principles of mathematics are few in number. If such were the case, deduction of the rest of mathematics could proceed from the few principles without further discussion or explanation. What we observe, however, contradicts this claim. In the *Principia Mathematica* of Russell and Whitehead, for example, after the primitive ideas and propositions are laid down and consequences are deduced, many pages of explanations are devoted in Section B (Theory of Apparent Variables) and on, and numerous definitions are given; but these explanations and definitions are and contain many hidden principles. The definiens in any of the definitions, for example, does not follow from the primitive ideas or propositions, otherwise it would be a theorem; hence it is posited as something new. Further, even if a definiens has only primitive ideas as elements, the manner in which these are composed is distinct from those elements; for the elements are to the composition as matter is to form. But form is distinct from matter; for example, the same piece of copper may assume various shapes, but shape neither follows from copper nor is reducible to or definable in terms of copper. Further, the definiens as a composite is posited as signifying something which will be used and hence as something which exists, otherwise it will signify a nonbeing, which is useless. It follows, then, that such definiens is a new principle because of its form and its existence. In short, hidden principles are very numerous in the *Principia Mathematica*, and they are found not only in definitions, as we have pointed out in Comm. 14 of B,1 of *Posterior Analytics*.

The other definitions of mathematics, too, are subject to objections, some of which are common to those against the Logistic School but others are proper to each of those definitions.

According to the School of Formalism, pure mathematics is the science of the formal structure of symbols and hence, indirectly, of the structure of objects. The terms "formal" and "without reference to meaning" are taken as synonyms, and so are the terms "a symbol" and "a mark with meaning"; accordingly, the formal structure or properties of symbols would be the formal structure or properties of symbols qua marks and not qua marks with meaning.

First, every discipline, whether a science or not, uses symbols; and symbols qua marks have structures. It would follow, then, that pure mathematics is indirectly applicable to every discipline; and this is false. This conclusion follows even if only some structures of symbols are investigated, namely, those which are simplest and which are regarded by this School as being the structures of numbers; for the simplest structures are more likely to be those which are common to the symbols of all disciplines, and these structures would be also the structures of the objects of those disciplines.

Second, let S be the set of structures of symbols qua marks, M be the corresponding set of structures of the mathematical objects, and C be the set of structures common to S and M. Let "S=M" mean that the sets S and M are identical, and "S>M" mean that the set S includes the set M, but not conversely. If S=M, there is no point in studying the symbols qua marks. One might as well study the mathematical objects directly and thus avoid duplication; for the aim of a mathematician has been and is the set M, and nothing is gained by the extra effort of showing — if possible at all, for there are certain difficulties — that S and M are identical. If M>S, the study of the symbols qua marks is inadequate for mathematics, for not all the structures or properties of the mathematical objects can be demonstrated. If S>M, one may leave out a part — the structures of marks which are regarded by this School as irrelevant — and attend only to the rest of S, which coincides with M. Then we are back to the identity S=M. We may add, the brevity of the symbols chosen is no argument in favor of getting to M through S, for, in studying M directly, the same symbols may be used as marks with meaning; and there is no gain in time by investigating S rather than M, for, psychologically, mathematicians use the same short-cuts in their thought processes whether they regard their symbols with or without meaning.

Third, if symbols are investigated qua marks, whatever is posited as a principle to be used as a premise cannot be true or false or a sentence or a proposition or an axiom, and two such principles cannot be consistent or inconsistent or contradictory or the like; for a sentence is true if and only if it is related in a certain way to the object it signifies, and two propositions are inconsistent if and only if they cannot be true at the same time, but symbols qua marks are not true or false or the like. One may assign different meanings to "truth" and "falsity" and other such terms if he wishes, but he would be doing this arbitrarily or dealing with different things. He may call $p.\sim p$ "inconsistent", but this word is in no way related to inconsistency as we understand it, and so it makes no difference whether he calls $p.\sim p$ "inconsistent" or "consistent" or "ridiculous" or by any other term; for all one observes in $p.\sim p$ is a set of marks, and marks as such have magnitude and shape and order and exist and the like but are not true or false or inconsistent in the usual sense. Further, $p.\sim p$ is avoided in mathematics because no two contradictories can be true at the same time; but $p.\sim p$ as marks exist at the same time and so are consistent in the usual sense of the word "consistent", and there is no reason for ruling them out as inconsistent unless they are considered as related to what they signify. Accordingly, the investigation of symbols qua marks cannot deal, even indirectly, with mathematical objects and their properties. As for the principles according to which marks may be operated, they are not truths but rules, which may be of two kinds: either arbitrary or dependent upon truths. For example, the rule according to which $x=b-a$ follows from $x+a=b$ is "When transposing a term, change its sign", and this rule depends on the axiom "If equals are added to or subtracted from equals, the results are equal". Thus such a rule is used because of a corresponding axiom, which is true; but that axiom is true because of what it states and not because of the rule. So the axiom is prior as a cause to the rule; for it causes the rule but is not caused by it. As for arbitrary rules, they may be changed at will; and they are rules for playing games and not rules from which truths can be investigated.

Fourth, no group of symbols has meaning if some or all of its symbols are regarded without meaning.

If all the symbols are regarded without meaning, what we have is a group of marks, and operations according to, or from, or on such marks are impossible unless those marks presuppose symbols which are meaningful. For example, let us consider the group "Two non-parallel lines intersect at just one point", and let this be abbreviated as "T n l i a j o p", in which each letter is the first letter of each word-symbol of the group (taken in the same order). If the word-symbols in the group are regarded without meaning, there is no difference between that group and "T n l i a j o p", except in brevity, for there is a one-to-one correspondence between the two. Evidently, if "T n l i a j o p" is taken as an axiom, operations according to it or from it or on it are impossible, unless axioms with meaning are presupposed; and even then, other difficulties arise, for differences between subjects, predicates, verbs, relations and the like do not exist in "T n l i a j o p", for the only differences observed are just differences in the letters.

Difficulties arise even if some of the symbols are regarded without meaning. Let us take as an example "Two L's cannot intersect at more than one p", where L and p are considered without meaning. Now intersection is a trait of some things but not of all things. One virtue does not intersect another virtue, and redness does not intersect greenness or a line or an action; and since intersection as an attribute is defined in terms of or is limited to the subject in which it is present (73a34-b3, 1028a32-6, 1030b14-1a14, 1045b30-2), only certain things can intersect one another, e.g., lines and surfaces and the like. Intersection is like equality and oddness; for oddness is indivisibility of a (natural) number into two equal numbers and belongs only to numbers, and equality is an attribute only of quantities, whether essential or indirect or derived. In regarding L and p as marks satisfying certain axioms — if these be axioms at all — in which they appear, one thinks that he is universalizing to the utmost, but such universality is only apparent. In fact, he is omitting to state the things which can satisfy the axioms, and in so doing he leaves out some knowledge. Further, without that knowledge the reader is not convinced of the truth of the axioms, but both the truth and its conviction are necessary for scientific knowledge (71b25-6, 72a37-b4). The fact that the things which intersect each other may not come under a genus which has a univocal meaning makes no difference, for things may be one by analogy also (1016b31-5), and such things should be indicated in a science. Then properties of these things can be demonstrated from principles which are one by analogy (76a37-b2, 98a20-9, 99a8-18). Thus if one wishes to regard L and p as variables, the limitations of their variability should be indicated just as x and y as variables in $y^2=x$ are limited to *numbers*, for y cannot be a virtue or a color or a man or the like.

The definition of mathematics according to the Intuitionist School is not as well formulated as the definitions of the Logistic and the Formalist Schools, but it may be expressed by a number of statements. Mathematics is founded on a basic intuition of the possibility of constructing a series of numbers or objects; it is thus founded on thought, not on symbolism or any particular language, which is only a means to thought; it is not timeless or static or dogmatic but growing and dynamic and fallible and always in process, and it can never be completely symbolized; and it is the product of social activity by fallible minds and so subject to revision and development.

The above definition is in one sense too general, but in another too limited; and although there is some truth in it, its terms are not so definite or accurate as to be limited and applicable to what is considered as mathematics.

It is generally admitted that, if one is to have mathematical knowledge, thought and intuition — if intuition is defined appropriately — are fundamental and prior as ends to symbolism or language, that mathematical knowledge is growing, that mathematical statements are sometimes erroneous and so what is considered as mathematical knowledge may be false, that mathematics is the product of social activity, and that some mathematical thoughts are subject to revision and development. But these requirements are not limited to mathematics; they are applicable to physics, to chemistry, in fact, to all sciences and arts. As requirements, then, they should be discussed in the nature of science or of any systematic inquiry or activity. But in what sense are they requirements? A definition of mathematics should be applicable to every instance of mathematical knowledge, and that knowledge, if in the form of a theorem, should depend on its principles in the sense that it follows from them deductively; but a theorem does not follow deductively from the fact that it is a product of social activity, except in an external way, so to say, otherwise one might as well include in the definition of mathematics such requirements as taking food, living, moving, sleeping, and many other activities without which mathematical knowledge is impossible. Further, not all instances of mathematical knowledge are subject to development; for such thoughts as "If A is greater than B, and B is greater than C, then A is greater than C", "The function x^2 is continuous", and "The prime natural numbers are infinite" are always true and are neither fallible nor subject to revision or development. Thoughts in mathematics which are fallible or subject to revision or vague or inaccurate are regarded not as mathematics in the main sense but as bad mathematics or as falsities and should be excluded; for, if not, the term "mathematics" may have many senses and so many definitions. Moreover, the definition does not make clear whether mathematics is the activity of thinking about mathematical objects or the corresponding thought which is the product of that activity or a combination or relation between the two. In general, a definition should be ideal or as accurate as possible, otherwise there is greater probability for vagueness, confusion, disagreement, error, and falsity. Evidently, the definitions given by the Logistic and Formalist Schools are better in this respect. Finally, only numbers as an infinite series are mentioned in the definition; but group theory and projective geometry and some other parts of mathematics are not reducible to numbers, and so they are not included in that definition.

Some remarks about other definitions may be made. According to Auguste Compte, mathematics may be defined as the science of indirect measurement; for the most striking measurements are not direct but indirect, as in the case of the determination of distances and sizes of the planets or of the atoms. This definition is too narrow and mentions applications rather than what is considered as pure mathematics, i.e., mathematics in the main sense. Measurement is only one of the many traits of mathematical objects or of objects which are treated mathematically, and indirect measurement is impossible without prior knowledge of direct measurement.

According to Benjamin Peirce, mathematics is the science which draws necessary conclusions. This definition is too wide; and it is subject to interpretation. First, it appears to deal with implication between propositions but does not restrict itself to the objects of mathematics as commonly understood, and so it seems to be close to the definition of logic — in the deductive sense. Second, if the phrase "necessary conclusions" means conclusions which are necessarily true, then the definition is limited either to a part of logic — for logic as generally understood includes also conclusions which follow from premises but need not be necessarily true — or to those conclusions in the sciences which are necessarily true; but if it means conclusions which necessarily follow from premises, then it amounts to the definition of deductive logic.

According to A. B. Kempe, mathematics is the science by which we investigate those characteristics of any subject matter of thought which are due to the conception that it consists of a number of differing or non-differing individuals and pluralities. This definition is too wide, for it applies to all the sciences.

Maxime Bôcher regards mathematics as follows. If we have a certain class of objects and a certain class of relations, and if the only questions which we investigate are whether ordered groups of those objects do or do not satisfy the relations, the results of the investigation are called mathematics. As in the case of A. B. Kempe, this definition is too wide, for it applies to all the sciences.

Finally, there are some who think that mathematics is concerned with certain concepts, mathematical concepts. Such doctrine runs contrary to what mathematicians are actually doing. How can one add or divide or multiply or take the square root of a concept? Further, concepts exist in the mind, but mathematicians are concerned with functions and curves and volumes and surfaces and the like, and these are not or need not be in the mind. No one would deny that mathematical concepts are necessary if mathematical knowledge is to be acquired; but such concepts are of the mathematical objects but are not *the* mathematical objects themselves.

4. Aristotle's definition of mathematics, and an analysis of it.

Before Aristotle's definition of mathematics is stated, the meaning of certain terms will be given for the sake of accuracy, and some remarks concerning his logic will be made. Some of the terms are introduced for convenience, but the meanings given to them are in Aristotle's works.

The term "subject" is to the term "trait" as a relative term is to its correlative, and that which is a trait is a thing which belongs to its subject either by being said of that subject or by being present in it (1a20-b9). If a trait is a genus or a differentia of a species, then we speak of it as being said of that species and is called "an essential element" of it, and if it is the nature of the species, it is still said of that species; but if it is not a genus or a differentia or the nature of a species, it is present in that species and is called "an attribute" of it. An attribute which belongs of necessity to its subject but to no other subject is said to be a property of it (102a18-22). An attribute which may or may not belong to a subject is called "an accident" of that subject. The term "belong" is used in a limited sense here. A is said to belong to B if it is either an essential element of B or the nature of B or an attribute of B. As in the *Posterior Analytics*, let "AB" mean that A belongs to every B, and "$\overline{\text{AB}}$" mean that no A belongs to B. A demonstration is a species of a proof; it is a proof of an attribute through the cause, i.e., a proof of a property. In a qualified sense, a demonstration is a proof of a necessary attribute through a part of the cause. For example, concurrence of the medians is a necessary attribute but not a property of a right triangle, and it is proved to belong to that triangle through its triangularity, which is a part of the right triangle. A theorem is a statement signifying that a demonstrable property belongs to a subject. A corollary is a statement signifying that a demonstrable attribute which is not a property belongs to a subject. If "AB" is true, A is said to *belong* to B; hence *belonging* is a genus of belonging.

Some examples may be given. Animality and rationality are essential elements of a man, being a function is an essential element of x^2, and being a group in algebra is an essential element of a field; concurrence of the angle bisectors is an attribute of a right triangle and also of a regular polygon, and $2x$ as the first derivative with respect to x is an attribute of x^2+5 and of x^2; maximum area with a given perimeter among plane figures is a property of a circle, and 6 as the third derivative of a function with respect to x is a property of x^3+ax^2+bx+c, but not of x^3; the statement "The function x^2 is integrable between limits" is a corollary; the statement "The perpendicular from the center of a circle to a chord bisects the chord" is a theorem; fever is an accident of a man. The statement "an equiangular triangle is an equilateral triangle" signifies a *belonging*.

A trait is to its subject as a part is to a whole, but not as a material part of that whole (for the term "part" has many senses, 1023b12-25), e.g., not as a hand is to the man whose hand it is. Evidently, the whiteness of snow is a part of snow in the above sense, for it is not the whole snow nor a material part of it but only a trait of it; sidedness is a part of a pentagon; bh as an area is a part of a rectangle with b and h as sides; and, in general, a genus or a differentia or an attribute is a part (in the above sense) of the subject to which it belongs.

Every mathematical statement in the form of a principle or a theorem or a corollary can be put into the form "every A is B" or "no A is B", in which B is a trait belonging or not belonging, respectively, to A as a subject. The forms "some A is B" and "some A is not B" may be used, but if mathematical theorems are to be proved through their causes — and a proof in a science should be *universal* and so through the cause, as explained in the *Posterior Analytics* — then these forms need not and perhaps should not appear or else should be changed to the previous forms, which are more definite. For example, the form "some functions have $2x$ as their derivative" should be changed to "every function of the type x^2+c has $2x$ as its derivative".

In mathematics, the word "is" in "every A is B" signifies not a temporal but a necessary unity of every A with B, and in "no A is B" it signifies a necessary denial of the unity of any A with B. Accordingly, B cannot be an accident of A. Further, B in the statement "every A is B" need not be a quantity, for quantities are not the only attributes or essential elements which belong to other quantities as their subjects. For example, straightness is a quality belonging to certain lines, parallelism is a relation and so "parallel to line AB" signifies an attribute which is a relation belonging to a subject, and "x^2 has $2x$ as its derivative" signifies the derivative $2x$ as being possessed by x^2 and hence as belonging to x^2. Thus the verb "is B" in "A is B" as used by Aristotle should not be interpreted grammatically, as is usually done by most modern logicians, for "is" (or "being" or "to be") has as many senses as there are categories, and a category may be a substance or a quality or a quantity or a relation or acting or being acted upon or a position or a possession or somewhere or sometime (1b25-7, 1017a7-27). Accordingly, the expressions "A intersects B", "A lies in B", "A contains B" and the like are, in Aristotle's system, of the form "A is B"; for intersection comes under the category of relation, lying comes under the category of whereness, and containing comes under the category of possession. Predication in a syllogism, too, has as many senses, and also many modes (48a40-49b9). For example, the statement "8 is a multiple of 4" follows syllogistically from "8 is double of 4" and "whatever is a double of 4 is a multiple of 4", and the predicates "double of 4" and "multiple of 4" are relations; and similarly with the other kinds of predicates.

What is regarded as a theory of relations might be considered by Aristotle to be, if at all, a science of relations; for, like quantity, relation is a category and a genus, and of a genus there is a science (87a38, 1003b19). But the syllogism for Aristotle is limited to what is common to the kinds and modes of predications having *universal* application, for it is in this way that *universality* can be achieved; and such limitation on syllogistic reasoning does not prevent the application of it to every science or every field where it is needed. If a mathematician includes the calculus of propositions or the theory of relations in his definition of mathematics because he must use them, then other scientists will have to include them in the definition of their own sciences for the same reason, and duplications as well as other difficulties will arise. Thus the problem of what is adequate or necessary in mathematics is a matter of philosophy and of scientific method and not just a matter of technique. Detailed discussion of these matters will not be given here; but it is sufficient to state that all statements in mathematical research are included in Aristotle's definition of a statement, and that Aristotle's logic and scientific method is presupposed by mathematics as defined by Aristotle and is adequate for the corresponding mathematical demonstrations, as I have shown by an example in Comm. 21 of B,11 of the *Posterior Analytics*.

Next, the definition of mathematics will be given and a discussion of it will follow.

D e f i n i t i o n : Mathematics is a theoretical science of the properties of essential quantities and of whatever belongs to such quantities, considered specifically or generically or analogically.

What is science? First, some traits of science will be considered. The term "science" may be used in a number of senses. In one sense, science is universal knowledge which includes both principles and the demonstration of theorems from principles. In a second sense, science is limited to the theorems as demonstrated from the principles. To have science in the second sense, of course, one must know the principles from which the theorems are demonstrated; so in both senses of the term, science as knowledge includes knowledge of principles. The term "science" is used also for what appears in a book, like a book on plane geometry, but science in this sense is for the sake of imparting science as knowledge to others; and there is a one-to-one correspondence between this sense and each of the other two senses of the term. The term has been used in other senses also. The important traits of science as knowledge will be considered.

Science is knowledge concerning things (or beings); and things exist, whether actually or potentially. So objects which cannot exist are not objects of science. By "object" I mean a being or a nonbeing, i.e., what exists or what cannot exist. Now it would be vain effort and of no value to try to investigate traits of nonbeing, i.e., of odd numbers which are also even or of a square with unequal diagonals or of a body travelling in two directions at the same time, for no trait (and traits are things) can belong to a nonbeing. Likewise, although denials of the form "P is not Q", where P is a nonbeing and Q is a trait of a thing, are true, they are excluded from science; for there is hardly any dignity or value in them, whereas science as generally understood is prized for its dignity and value. Further, if in a science one were to posit P without being sure whether P exists or not and then prove the statement "P is Q", where Q is a trait, he could not be sure whether the statement proved were demonstrated from truths or not; so the dignity or value of "P is Q" in that science would likewise remain in doubt. Accordingly, such a statement would not be scientific in the main sense but only hypothetically.

The objects of a science come under one genus of things or under one aim. Botany has plants as its objects, statics has bodies at rest under forces as its objects, medicine has health as its aim, and strategy has victory as its aim. Subordination of one science under another is possible, and there are kinds of subordination. Botany comes under biology, plane geometry comes under geometry, the science of cancer comes under medicine, and mathematical physics comes under mathematics as an application. In some cases of subordination, the term for the higher science has two senses. By "geometry" one may mean knowledge of magnitudes taken generically only, or he may mean knowledge of magnitudes taken both generically and specifically, in which case plane geometry would be included. Similarly, the term "science" may be limited to *knowledge* of the properties of a genus qua that genus, or it may be used comprehensively to include also *knowledge* of the properties of any species or trait which comes under that genus. In a very special case, it may be limited to *knowledge* of a single property, e.g., of the concurrence of the medians of a triangle. When we speak of mathematics as a science, we usually mean science in the comprehensive sense.

Science is universal; for science is knowledge not of an individual thing or fact existing at a certain time or place or in some manner but of any thing or fact of a certain kind regardless of where or when or how it exists. Evidently, the concepts and axioms and definitions and hypotheses in science are universal, for what is signified by "straightness" or "function" or "three-sided figure" or "a line may be drawn between two points" is not limited to an individual thing but extends to every thing of a certain kind. Theorems, too, are universal, as in the theorem "The angle bisectors of a triangle are concurrent"; for, if the principles as premises are universal, the conclusions, which are theorems in a science, are universal also. Further, there is a reason for universality. Differences in time or place or manner in individuals do not contribute to the nature of those individuals or the properties of that nature, and so they cannot be causes of that nature or its properties. But is there no science of the Moon or of the Sun? One may use the term "science" in a limited sense also so as to be applied to an individual, but such application presupposes science as universal knowledge; so this sense of the term is secondary and need not be considered here.

Since science as knowledge includes elementary concepts and combinations of these (e.g., in elementary geometry one should have the concept of straightness and of a triangle and the like and such *thoughts* as of the fact that the area of a circle is πr^2 and of the fact that vertical angles are equal), the general traits of such concepts and their combinations would be traits of science. In the case of concepts, if they are elementary, they must be clear of the things conceived; but if not elementary, they must be had in the form of a definition whose parts are not further analyzable, as in the case of a triangle; for one must have a definition of it, otherwise the attributes proved of it cannot all be properties. As for *thoughts*, they are axioms or hypotheses or theorems demonstrated from principles; and since in a science each of them signifies a universal fact, i.e., what belongs or does not belong to a subject universally, it must be true; for a true *thought* is one which signifies a fact. Thus clarity of elementary concepts, nonanalyzable definitions, and truth are general traits of science.

In science, since conclusions are demonstrated from principles by the use of logic, knowledge of logic and of the use of it is necessary. But logic is not a part of mathematics; it is used by or is applicable to mathematics and other sciences, and in this sense it is more universal. A demonstrated theorem is a theorem along with the demonstration of it.

Whether the general methods of acquiring the principles of science should be included in the definition of science or not, e.g., such methods as sensation, abstraction, induction, experiment, and the like, is a matter of choice; and the same applies to intuitions, i.e., to the possessions of or abilities to acquire those principles. In either case, scientific knowledge is impossible without (*a*) principles or (*b*) the methods used to acquire principles or (*c*) the corresponding possessions of or abilities to acquire those principles. It may be added, Aristotle's term *νοῦς*, translated as "intuition" here, should not be confused with "intuition" as is vaguely and popularly used. We might have used some other term, such as "conception" or "insight" or "intellect" or "apprehension" or the like, but once the distinctions have been pointed out, a scientist qua scientist should not be concerned with popular terminology or public relations.

With the above traits of science as a basis, a definition of a science in the comprehensive sense may be given.

Definition: A science is universal knowledge of things and facts under one genus or with one aim, and it consists of (*a*) principles, which are concepts and definitions and axioms and hypotheses acquired by various methods (sensing, abstracting, analogy, intuiting, inducing, etc.), and of (*b*) demonstrated theorems from principles by the use of logic. It is assumed that the axioms and the hypotheses are true and that the concepts and the definitions are of things.

With respect to aim, there are three kinds of sciences: theoretical, productive, and practical. The aim of a productive science is the production of something, e.g., of a house or of health, and the corresponding science is architecture or medicine, respectively. The aim of a practical science is *action*. Politics is such a science, and virtuous *action*, which contributes to the happiness of the citizens of a state, is its aim. Both productive and practical sciences presuppose knowledge; for both the architect and the statesman should have or acquire the knowledge necessary to achieve their aims. Finally, the aim of a theoretical science is truth for its own sake and not for the sake of its application, and mathematics, often called "pure mathematics", is a theoretical science. Mathematics has sometimes been called "useless", but what is meant is that the activity of the mathematician has as its aim the discovery of truth not for the sake of use or application but for its own sake. Mathematics may be used, but the use of it is the concern of scientists who apply it to their own fields, e.g., to mathematical physics or engineering or chemistry or other sciences which use mathematics.

The definition of mathematics mentions only essential quantities. How do derived quantities come in? They are included in the phrase "of whatever belongs to such quantities". For derived quantities are attributes of essential quantities and so belong ultimately to essential quantities; hence a property of a derived quantity is a property of something which belongs ultimately to an essential quantity. For example, a property of certain angles belongs ultimately to certain surfaces or lines (depending on how one defines an angle).

How do indirect quantities come in? They are composites having quantities as attributes but physical things as subjects, using the term "physical" in Aristotle's sense, which is wide. For example, the term "five" in the expression "five seconds" signifies an essential quantity as an attribute of a subject, but the term "seconds" signifies a physical subject (i.e., time) to which that attribute belongs; and since five as an attribute belongs to other things besides seconds, no property of five can be a property of five seconds, and conversely. Similar arguments apply to every indirect quantity. Universally, then, no property of an essential quantity or of whatever belongs to it can be a property of an indirect quantity; so indirect quantities and their properties cannot come under mathematics, for they are excluded by the definition of mathematics. However, essential quantities and things which belong to them are applicable to indirect quantities; but the nature and kinds of application and the manner in which quantities are applicable will not be considered at present. In a similar way, accidental quantities cannot come under mathematics.

Why should mathematics limit itself to the proof of attributes which are properties? It was stated earlier that a property of a subject cannot belong to any other subject. Let P be a property of the subject S, and let the species of S be X, Y, and Z. Then P is an attribute but not a property of X. If we let D be the differentia of X, then PX can be proved without the use of D in the proof; for, as a property of S, P can be demonstrated to belong to S, but S has no D in its definition. Consequently, D cannot be the cause or part of the cause of PX, and the proof of PX is a corollary, or else a demonstration in a qualified or secondary sense (78a22-b31), but this is not a demonstration in the main sense, as we defined it. Further, the proof of PX is not *universal*; for P belongs to Y and to Z also.

In modern mathematical terms, the proof of PX is not the most general, and a desirable mark of a mathematical theorem is maximum generality or necessary and sufficient conditions. A theorem as defined, then, will require two parts for its complete proof. Let M, N, and P be the immediate genus, the differentia, and a property of S, respectively. Then one of the parts will be the demonstration of PS; the other will be the proof of NM from PM, i.e., the proof of the statement "If P belongs to M, then N belongs to M" or "If P belongs to M, then M is S". To use a mathematical expression, the demonstration of a property amounts to the proof of the statement "The necessary and sufficient condition for M to have N is that M have P". The two parts of such a theorem give rise to the philosophical problem whether a thing can have more than one definition. According to Aristotle, a thing can have only one definition, and this uniqueness rests on a certain priority; according to almost all modern scientists, a thing may have more than one definition. This problem will not be considered here.

Examples of theorems may be given. An equilateral triangle is equiangular, and conversely. Aristotle considers "equilateral" as signifying a differentia but "equiangular" as signifying a property of an equilateral triangle. The equilateral is defined in terms of sides but the equiangular is defined in terms of angles, and a side is prior in definition to the angle; for an angle is defined in terms of sides, but not conversely. Again, in a plane, the equation of a curve is the equation of a straight line if and only if it is of the form $Ax+By+C=0$, with A and B not both zero; the curve whose equation is $y=ax^2+bx+c$, with $a\neq0$, cuts the x-axis at two points if and only if $b^2-4ac>0$; a function is integrable if and only if its points of discontinuity are of zero measure. Is $2x$ as a first derivative a property of x^2+c? It would appear so. On the other hand, nx^{n-1} as a first derivative of x^n+c where n is a natural number, is more general, and there appears to be greater generality if n is a rational number; so $2x$ as a first derivative of x^2+c would appear to be a corollary or a special case or an application of a theorem. There are problems concerning the nature of a theorem. Or is this a matter of definition?

In practice, what is given as a theorem is often a corollary and not a theorem. If it is a theorem, sometimes one part of it is proved only, and the other part may be difficult to prove or is implied or is left out for some reason, e.g., the inadvisability of introducing it to young students. In a plane, the two tangents to a circle from an external point are equal, and this equality is a property of a circle; but it is inadvisable in a beginner's course to include in the theorem the proof that if two tangents to a closed figure from an external point are equal, then the figure is a circle. If what is given as a theorem is not a theorem but a corollary, it is often difficult to demonstrate the corresponding theorem, which is more general, and sometimes the mathematician himself is not aware that his proof is of an attribute but not of a property. Such was the case of integrability as an attribute of a subject; for its proof as a property was a recent discovery. Such situation raises a question.

If one proves an attribute of a subject without knowing that the attribute is not a property, what does he know? Now a man's scientific knowledge, e.g., that P is a demonstrated property of S, includes his conviction of the fact that he knows S to be *the* cause of P. If a man proves PX, where X is a species of S, and thinks that the whole of X causes P and hence that he demonstrates PX, then he is mistaken; for the differentia D of X is irrelevant to the demonstration of PX, and he who thinks that the irrelevant is relevant (i.e., that it is the cause or part of the cause of P) does not know the cause and is therefore mistaken. We may add, one is more likely to make this mistake if he happens to use D in the demonstration of PX, as I have pointed out in Comm. 5 of A, 5, of the *Posterior Analytics*. And if a man proves PX without the thought of whether X is or is not the cause of P, he does not know the cause, even if X happens to be the cause.

Further, if he knows that he is using only a part of the cause but cannot specify that part, he still does not know the cause. In all cases, he does not know the cause; in the first case, his ignorance arises from a false *thought*, in the second, from no *thought* about the cause (79b23-4), and in the third, from the *thought* that he cannot specify the cause. In all cases he knows the fact PX as proved, and such knowledge is true; but he does not know the cause of that fact, and such knowledge is not through the cause and so is not scientific — he does not know what is relevant and what is irrelevant in the proof of PX. So in one sense he knows, for he knows the fact as proved, in another sense he does not know, for he does not know the fact as being demonstrated.

What kinds of things in mathematics may be properties of subjects? Russell's criticism of mathematics as a science of quantity rests on a number of misconceptions, two of which will be pointed out. First, he identifies the term "quantity", as used by his predecessors, with the term "number". But it is clearly stated that magnitudes, too, are quantities, for they are species of quantities (4b20-5a37), and hence that geometry, which is concerned with magnitudes (1143a3-4, 1355b30-1), comes under mathematics. Second, he misrepresents the ancient definition; for the phrase "science of quantity" mentions quantity as the subject of mathematics but leaves out the kinds of things which may be properties of that subject. But it is clearly stated that a science — and hence mathematics — is concerned with certain attributes of its subject (75a39-b2, 76b11-16), and that those attributes must be properties of those subjects (73a21-74b4); for only properties are proved through the cause, and scientific knowledge is acquired only through demonstration and hence only through the cause (71b9-12). Now properties are attributes and are therefore present in their subjects; hence they need not come under the same category as that of their subjects. For example, such things as parallelism, intersection, concurrence, inclusion, separation, betweenness, and relative position are attributes of certain quantities, and none of them is a quantity; yet all of them may be properties of certain quantities as subjects (75a39-b2, 76b11-16, 1061a28-35). So it is evident that projective and descriptive geometry, which have nothing to do with numbers, come under mathematics; for their subjects are magnitudes, whose properties need not be numbers. In fact, numbers (or *numbers*) as properties of magnitudes are derived quantities, and most properties of magnitudes are not numbers. Further, since the terms "subject" and "attribute" are relative to each other, and since the same thing may be an attribute relative to one thing as a subject but a subject relative to another thing as an attribute, mathematics includes theorems in which the subjects, although attributes of quantities, are not themselves quantities. For example, separation and betweenness are attributes of points on a line, but they are subjects of their own attributes; for there are postulates concerning betweenness and separation as subjects, and attributes can be proved from these postulates. This is true in other sciences also. Virtue is an attribute of man; but it is a subject of its own attributes.

Not all proofs deal with properties of subjects; for, in some cases, what is proved is the existence of certain composite subjects (76a31-36). For example, the existence of a triangle, which is a composite of a certain kind, can be proved by construction (92b15-6). Thus some things in mathematics are posited to exist, others are proved to exist or be possible (1051a21-33). As for proofs about attributes which are not properties of subjects, their inclusion in mathematics is a matter of choice. If they are to be included, then the term "properties" in the definition of mathematics should be replaced by the term "attributes"; but if proofs of attributes are to be limited to demonstrations, which show the cause, then the term "properties" should be retained. In the former case, mathematics is defined as it is actually presented, with its shortcomings; in the latter, it is defined as it should be, as an ideal to be attained.

The phrase "considered generically or specifically or analogically" in the definition of mathematics takes care of the various parts or branches of mathematics.

Now no property of a subject is a property of a genus or of a species of that subject. For example, a property of a triangle is not a property of a genus of a triangle, e.g., of a polygon; nor is it a property of a species of a triangle, e.g., of a right triangle. Hence no theorem in the science of triangles qua triangles is a theorem in the science of polygons qua polygons or in the science of right triangles qua right triangles. But since a trait of a genus belongs to a species of that genus, a property of a genus is an attribute of a species of that genus. Consequently, with respect to the order of scientific learning, the science of a genus is prior to the science of a species of that genus; and he who investigates the properties of a species should know the needed properties of all its genera, thus avoiding duplication of proofs. For example, the science of polygons should precede the science of triangles; the science of lines should precede the science of straight lines; the science of groups should precede the science of abelian groups or the science of fields; and the science of the affine group should precede the science of the Euclidean subgroup.

Are there exceptions? If a property of a subject is a relation to another subject, the corresponding theorem may be so rephrased as to be a theorem concerning the latter subject. For example, a square of side $2s$ inscribed in a circle of radius r leads to $2s = r\sqrt{2}$; so a circle of radius r and circumscribed about a square of side $2s$ leads to $r\sqrt{2} = 2s$. In general, if A has the property RB, where RB is the relation of R to B, then B has the property $\bar{R}$A, where $\bar{R}$ is the converse relation of R to A. Linguistically, the two theorems appear different, but they amount to the same theorem, so only one theorem may be given; and if both theorems are given, they still come under the definition of mathematics, even if they amount to the same thing. Perhaps there are other such exceptions.

Some properties must be stated analogously, for they are properties of subjects which are one by analogy (1016b31-35). For example, lines and numbers are called "quantities", and it would appear that the genus "quantity" is univocally predicable of a line and of a number; but this is not the case. Numbers cannot be compared with lines, i.e., they are neither equal nor unequal to lines, and so they cannot be added to or subtracted from lines; for what would the remainder be if three lines were subtracted from five numbers? Likewise, lines cannot be compared with surfaces or with solids or with angles. But there are statements — some of them principles and others theorems — which are true within each species taken separately. It is in this manner that the statement "Sums of equals are equal" is true for numbers, and for lines, and for angles. The things which are equal must all be comparable and so of one kind or of one species, e.g., they must all be numbers, or all lines. Consequently, there are principles each of which is one by analogy (76a37-b2), and a theorem which follows from such principles is likewise one by analogy (99a15-8) and has a cause which is one by analogy;

and such principles and the theorems which follow from them are analogously applicable to the various species or kinds. Of this kind, too, are such principles as the commutative law, the associative law, the distributive law, and the like. These are posited nowadays as mathematical axioms of a general sort, and no mention is made as to their nature and their applicability; but there is a problem.

Let us consider the commutative law, e.g., the law $p+q=q+p$. Is it a mathematical axiom? If p and q are known *universally* and the subject of mathematics is posited, the question can be answered; but if p and q are posited as symbols without reference to meaning but subject only to rules, the objections stated earlier against such position may be repeated here. Now the commutative law is true of certain things which are considered mathematical, e.g., of angles and real numbers and vectors, but it is true of other things also. If A is a partner of B, then B is a partner of A, and an example of the plus-symbol is "is a partner of"; similarly, the expressions "is different from" and "may be combined with" may be given as examples of the plus-symbol in the commutative law. So the plus-symbol applies to economics, to philosophy, to chemistry, and to other disciplines. Perhaps a more general form of the law would be $p\mathrm{R}q=q\mathrm{R}p$, where R signifies a relation — for an operation is an instance of a relation — and the equality-symbol may signify equality as well as other things, some of which are mathematical but others extend beyond mathematics. So the law, taken *universally*, is applicable to many sciences; and perhaps it belongs to the science of relations. Similar remarks may be made concerning other laws which are used by mathematicians.

There are many axioms which are one by analogy and are used by more than one science, the principle of contradiction being one of them. Accordingly, a mathematician may use axioms by analogy, and these may belong to mathematics or may extend beyond mathematics. Now from the technical point of view of proving theorems it makes no difference whether the mathematician knows or not that an axiom which he posits and uses is mathematical or extends beyond mathematics, as long as his proof is a demonstration, and it makes no difference whether he knows or not that the axiom is not univocal but one by analogy, as long as he does not misapply it. But if he wishes to consider the nature of that axiom and its relation to mathematics as a science, he needs principles which differ from those of demonstrating mathematical theorems; and such principles belong to philosophy and to the science of demonstration.

Finally, one might raise the problem whether the definition of mathematics as a theoretical science should be given in terms of its subject or in terms of method, and whether mathematics is limited as a growing science if it specifies its subject. The two problems are not limited to mathematics but are universal in nature. If a body of theoretical knowledge is to be regarded as one science and not many, then it must have some kind of unity; so if mathematics is to be one science and not two or three or some other number, its definition must indicate the principle according to which it is one and not many. Is that principle one method or a set of methods or one subject?

First, the various methods are common to all or almost all theoretical sciences, whether directly or indirectly, although the degree to which each of them is used varies from one science to another. Dialectics, sensation, abstraction, induction, proof, experiment, hypothesis, and the rest of the methods are used by every science, so the principle of unity cannot be a method or a set of methods. We have shown the difficulties faced by the Logical and Formalist Schools which tend to place emphasis on method as the principle of unity in mathematics. Almost all, if not all, existing theoretical sciences consider the subject as the principle of unity, and no other principle has been put forward. Biology is concerned with animals, astronomy with the heavenly bodies, physics with movable bodies qua movable, geometry with magnitudes, and similarly with other theoretical sciences.

Again, a method is to that to which it belongs as an attribute is to its subject, and an essential attribute cannot be defined or be understood without the subject to which it belongs (73a34-b3, 1028a10-36). For example, oddness is an essential trait of number though not of every number, and it is defined as indivisibility of a number into two equal numbers. Thus if a method were limited to one subject, it would be definable in terms of that subject, and not conversely, but if it were a method of many subjects, mathematics could not be defined in terms of that method. The same may be said if a set of methods were limited to one subject. Besides, if a method or a set of methods were limited to one subject, one could define the corresponding science in terms of that subject also; and the problem which would arise would be one of priority, namely, whether there is a priority in definition of a subject over a method or a set of methods. This is a metaphysical problem, discussed in the *Metaphysics*, Book Z, in which Aristotle concludes that the subject is prior in definition to an essential attribute of it. It follows, too, that a definition in terms of attributes is unacceptable; for, besides the priority in definition of a genus over its essential attributes, these are many, but the genus under which they come is one and functions as their unity.

From the above arguments, then, it appears that the principle of unity in a theoretical science, and hence in mathematics, is one subject and not one method or a set of methods.

Second, the definition of a theoretical science in terms of one subject is no bar to growth of that science. This is evident from what follows.

Generically, the theoretical sciences are three: first philosophy, mathematics, and physics (1064b1-3), assuming that each of them is comprehensively taken, e.g., if mathematics is considered as defined previously. Specifically, on the other hand, they are perhaps infinite; for an ultimate genus has many species under it, the properties of a genus differ from those of a species of that genus, and the number of such properties is perhaps infinite (170a22-3). For example, the properties of quantities differ from those of magnitudes, those of magnitudes differ from those of surfaces, and the properties of each geometrical curve or surface — and the kinds of such curves and surfaces are certainly a great many — differ from those of any other kind of curve or surface. Further, each subject under quantity has perhaps an unlimited number of properties; for a property may be a relation, and a thing may be related to many other things in perhaps an infinity of ways. Again, according to the definition of mathematics given earlier, the properties investigated need not be of a quantity only; they may be properties of whatever belongs to a given quantity, and such properties need not be quantities, as stated earlier. The roots of the second degree equation, i.e., of $Ax^2+Bx+C=0$ with $A\neq0$, are a property of that equation, but they themselves have properties, e.g., the property that they are equal if and only if $B^2—4AC=0$; and similarly in the case of equations of other degrees, generically or specifically taken. In the same way, separation, betweenness, inclusion, intersection, mathematical correspondences and the like, which are not quantities but belong to quantities, have their own properties.

The problem concerning growth may be viewed from the kinds of expressions in a science also. The principles in a theoretical science are indefinable terms, definitions, axioms, and hypotheses; and from these the scientist demonstrates theorems. One of the terms signifies the subject of that science as a genus, e.g., quantity in the case of mathematics. Assuming that the subject in that science is not changed, growth is possible with respect to all the kinds of principles and with respect to theorems. One may introduce new indefinable terms, or new definitions in terms of indefinable terms already posited, or new axioms, or new hypotheses; and one may demonstrate theorems indefinitely.

Again, a subject has its own proper principles. If A and B are neither species nor attributes coming under a genus, their proper principles cannot be combined to form premises for a conclusion; for the terms in A's principles exclude those in B's principles. Accordingly, (*a*) if the proper principles in the science of B are not axioms in the science of A, they cannot be so used as to increase the theorems in the science of A, and (*b*) if they are axioms in the science of A (like the proper principles of logic, which are axioms in mathematics), again the theorems in sciences A cannot be increased, for they are already in the science of A. It follows, then, that if one were to combine A and B and form a science of A and B, (*a*) the possible theorems in the science of A and B would be the same as the possible theorems in the science of A and those in the science of B, and (*b*) such combination will be of no help to the growth of the science of A or of the science of B. Each of these sciences, then, has its own unity, and to try to combine them is like trying to combine two men and call the result one man (75a38-b20, 77a26-31).

It is evident from the above, then, that while the subject of a theoretical science remains the same, the growth of that science is unlimited. Further, such a science should have a definite subject and should include it in its definition and not leave us in the dark as to what kinds of things are to be investigated. If one has no clear idea of what kinds of things he is investigating, how will he know whether his discoveries are mathematical or psychological or geological or what not, and how will he know whether the principles he has used are mathematical or political or biological? Relations between sciences certainly exist, e.g., between mathematics and physics, and between physics and politics, but such relations are the subject of the nature of science in general and not of a particular science. The Logical and Formalist Schools were more definite about the subject of mathematics than the Intuitionist School.

5. The relation of Aristotle's definition to modern mathematical research.

We may now turn to Aristotle's definition of mathematics in its relation to modern mathematical research. Let it be assumed that the three main parts of modern mathematical research are (modern) analysis, algebra, and geometry; and let their respective percentage ratios be, roughly speaking, 70:20:10. Topology is included, and let it be regarded as a part of analysis. How much of this research comes under Aristotle's definition of mathematics? Some preliminary remarks of a general nature will be made and some philosophical difficulties faced by modern mathematicians will be pointed out, for a mathematician must make — and he often does make — assumptions which are not mathematical but are presupposed by mathematics.

Since mathematical objects are things (impossible objects are excluded from mathematics or any science, 100a9), all properties of things qua things are attributes but not properties of mathematical objects. For example, definitions must appear in mathematics, and these must be posited in accordance with the nature and properties of a definition; and demonstrations by means of *reductio ad absurdum* are sometimes used, and these depend on the principle of contradiction. But discussions of the principle of contradiction and of the nature of definition belong to philosophy, which is the most universal science, for all other sciences use definitions and assume the principle of contradiction. Similarly, the mathematician must assume the truths of logic, for demonstrations are syllogisms, and these belong to logic. But the problem of whether the premises of mathematics are relatively few or not belongs to logic and not to mathematics. The properties of a demonstration, too, must be assumed by the mathematician, for a demonstration, being *universal*, must show the cause and so exclude what is irrelevant in proving a property. But a demonstration is defined and discussed in the nature of science (*Posterior Analytics*, for Aristotle) and not in mathematics. Again, a mathematician uses terms, statements, and other expressions, and these are acquired by certain powers of the soul (or mind, if you wish) and exist in a certain manner. But the discussion of the powers used and of the manner in which things exist in the soul belongs to psychology and not to mathematics. It is evident, then, that whether intuition is necessary in mathematics or not, whether the meaning of undefined terms are determined by the axioms posited for them or not, whether the concepts and the axioms of mathematics are relatively few or not, whether mathematics and logic are the same or not, and whether mathematical assumptions must be true or not are not mathematical problems but belong to other sciences. A mathematician may discuss these and take a definite position, but unless he is competent in the corresponding sciences, his position will be that of an amateur.

The principles laid down for modern mathematics usually fail in many ways. For example, it is held that analysis rests on the principles of the *number* system, and that whole *numbers*, fractions, rational *numbers*, real *numbers* and all others can be defined in terms of those principles, i.e., in terms of units or objects or members, a set or a class or a collection, and certain attributes or relations between these, such as membership and the like, all of which are principles as elements. Some thinkers require the elements to be of the same kind. It thus appears that what is regarded as continuous is thought to be definable in terms of certain primary things. When it comes to the meaning of the term "element" or "object" or "member", which is taken to belong to a class or set, it is usually assumed or taken as understood or left open; and examples offered as elements are usually substances, e.g., men or marbles, but sometimes they are attributes, e.g., letters or colors. Again, in view of the necessity of including zero and negative *numbers*, some thinkers merely posit these as existing, but others so define them as to raise doubt of their existence; and some identify positive integers with natural *numbers*, while others, like Russell and Whitehead, speak more philosophically and distinguish the two kinds.

Such presuppositions give rise to a number of philosophical difficulties, some of which will be mentioned. In the *number* system which posits objects or members as elements, it is necessary for an element to be either any object, or a specific object, or what is common to all objects.

First, if it is any object, let it be a man. Then the things which are not elements but are defined in terms of elements will have to be defined in terms of man or men, and such things as a fraction and a negative *number* and $\sqrt{2}$ will have to be defined in terms of men. But no one observes a man in $\sqrt{2}$, as $\sqrt{2}$ is commonly understood. So neither a man nor an animal nor any substance can be an element. Similarly, no quality or relation or action or any category which is not a quantity can be an element. Second, if the element posited is a specific object, since such object cannot be in any of the categories just considered, it will have to be under the category of quantity; and such an element may be a point or a line or a number or a unit or an attribute of any of these. But of those who posit objects or members as elements without specifying those elements, some regard points and lines and the like as defineable in terms of the elements. However, those who mention points and lines and, in general, mathematical objects as elements are not subject to the same criticism. Third, if the objects posited as elements are abstract or common to all the categories, they will be, at best, elements of philosophical objects and hence of all things, and mathematics will be identical with philosophy, a position held by the Pythagoreans and also Plato in his later theory; and the contradictions faced by such position may be repeated (see Aristotle's *Metaphysics*, Books A, M, and N). Further, if mathematical objects can be defined in terms of such elements alone, so can the objects of biology and of physics and of the rest, and there will be only one science, such as Plato's dialectics; but this is impossible. Again, it is impossible to define the less universal in terms of the more universal alone, e.g., man in terms of animal alone, for the less universal needs another principle, and this is a differentia (998b30-1, 1057b22-3). Accordingly, the objects proper to mathematics, e.g., points and lines and surfaces, will require in their definition or their understanding other principles besides the elements common to all things.

Whole *numbers* and fractions and irrational *numbers*, too, cannot be defined in terms of classes, their elements, and their attributes or relations alone; for if they were, they would not be applicable to mathematical objects as these exist and are usually understood. Five marbles as something existing and as commonly understood is not a certain class of classes; the diagonal of a square with a unit side is one thing and has a last point and is not an instance of a set of pairs (a, b) with $a^2 < 2b^2$ and with no last or greatest (a, b); and distance or a surface as it exists is not reducible to or definable in terms of classes and elements and their attributes and relations, all universally taken. One may introduce artificial definitions and assert that the properties which follow from them are in a one-to-one correspondence with those which are commonly or popularly accepted of the mathematical objects as these exist and are understood, but correspondence is not identity; for any correspondence between A and B presupposes some difference between A and B, and such difference necessitates properties of A which are not properties of B, and vice versa. A one-to-one correspondence between a circle and an ellipse exists, but an attribute of a circle is not necessarily an attribute of an ellipse; for a property of a circle belongs to a circle alone. If one is interested only in common attributes, so be it; but proper attributes, which are properties, exist also, and these cannot be excluded from mathematics. For example, the properties of a circle are parts of mathematics. Finally, if one excludes such specific properties as parts of mathematics, he will have to exclude also most of what he considers to be common attributes; for most of the latter are specific relative to attributes which are the most common attributes of all things. If so, we would be dealing with the properties of things qua things and not with mathematical properties, and our science would be philosophy and not mathematics. In short, we would be commiting the errors of Plato and the Pythagoreans, who reduced everything to the principles of one science.

If we assume that the foundations of analysis as presented are inapplicable to some mathematical objects as these exist and are understood, does it follow that all research in that field is false or useless or meaningless? Certainly not. True conclusions follow from true as well as from false premises; and those who are engaged in laying down the foundations of mathematics frame their principles in such a way as to arrive at conclusions which, on the verbal level, are generally accepted as true; and what follows from these conclusions is clear sailing. This happens in physics, too, where alternative hypotheses lead to the same results. But alternative hypotheses are contradictory, and either all are false or only one is true; and this fact does not change just because they all "work". A physicist or a chemist might be somewhat content with such hypotheses, because of certain difficulties in his subject matter; but a mathematician cannot afford to be so disposed, because such difficulties do not exist in the subject of mathematics.

The commutative law of multiplication, i.e., $A \times B = B \times A$, may be used as an illustration. It is usually taken as an axiom by most thinkers. The "A" in that law, taken as an axiom, is regarded by the mathematician as having the same meaning in "A×B" as in "B×A", and for Bertrand Russell, too, who regards the law as a theorem, A is taken as being the same. For Aristotle, however, "A" does not have the same meaning for the two products, whether in arithmetic or in geometry. First, its meaning in arithmetic will be examined.

Let us consider a business man who has 8 boxes, each of which contains 12 hats, and also 12 boxes, each of which contains 8 hats. Knowing that 8×12 is equal to 96 in the first case, he concludes mathematically by the use of the commutative law that 12×8 in the second case is 96 also; and then he attaches the number 96 to the hats and says that there are 96 hats in each of the two sets. But there is a difference between 8 boxes in the first case and 8 hats in the second, for 8 in the first case has a box as a unit, but in the second case it has a hat as a unit; and the same applies to the number 12 in the two cases. Evidently, there are two kinds of units in the multiplication. If the mathematician is not interested in philosophical matters such as units and their nature, he can have no claim to laying down the foundations of mathematics; for such claim is a matter of philosophy or the philosophy of mathematics, in which the discussion of units is included (253b2-6, 1052a15-1054a19, 1061b18-27).

In his *Introduction to Mathematical Philosophy*, Russell treats multiplication more philosophically, for he specifies somewhat the members of a class; but his definition of multiplication is artificial and is faced with difficulties. According to him, if class A has as members the three objects p, q, and r, and class B has as members the two objects x and y, then the product A×B may be defined as the class which has as members the six couples (p, x), (p, y), (q, x), (q, y), (r, x), (r, y). Russell wishes to get 6 objects as the product of 2 and 3, and he so frames his definition as to get 6 objects; but what kind of objects does he get? First, the members of A and of B appear as elements, but the members of A×B and of B×A are different, for they are pairs, and this assumption is arbitrarily introduced. Second, he makes an assumption when he posits a pair (a, b) to be different from the pair (b, a). Third, there is the problem as to what is meant by "a pair", for it can have many meanings. Fourth, the multiplication as defined cannot be applied to the multiplication as usually understood and made use of. If we let A be 8 and B be 12, as in the case of the boxes with the hats, each member of 12 will be a hat, and so will each member of 8×12; but Russell would make each element of 8×12 be a (box, hat), which is ridiculous, or meaningless, or else false.

In arithmetic, which is concerned with numbers, Aristotle starts with what people actually do and then makes an abstraction from the things under consideration to obtain universality, thus retaining applicability; for, in abstracting P, which is an attribute or an essential element of or a fact about some whole Q, he leaves aside all else which belongs to Q and attends to P only, thus making possible the investigation of properties of or other facts about P in a universal manner. Evidently, P and its properties will still belong to Q, and they belong to other things also. Now in $A \times B$, B is not just a number but a number of units of the same kind, e.g., a number of feet or of lines or of men or even of numbers, and the symbol "A" signifies a number of units each of which is a B and not a unit of B; and as for $A \times B$ itself, it is the total number of units each of which is the same in kind as a unit in B. In the case of the boxes with the hats, 6 in 6×8 is the number of boxes, and a box with 8 hats is the unit; 8 is the number of units in each box, and a hat is the unit; and 6×8 is the total number of hats, with a hat as the unit. Evidently, "6×8" does not have the same meaning as "8×6", for in the latter case each box has 6 hats and not 8 hats; but the total number of hats in each case is the same. Thus the equality $6 \times 8 = 8 \times 6$ indicates both a sameness and a difference; it is not like the equality $48 = 48$, which is an identity. The difference lies in the manner in which the 48 units are obtained in the two cases, the sameness lies in the equality of the number of units reached in the two cases; and the axiom states that, in spite of the difference in the manner, the two numbers reached are equal, for equality is sameness in quantity, and a number is a quantity.

In Aristotle's geometry, the formulation of the above law as far as its applicability is concerned, although not as simple, would be similar to that in arithmetic; and the law, taken universally, would be applicable to numbers and to magnitudes by analogy (76a37-b2). Even within geometry, it would be applicable by analogy; for there are many kinds of magnitudes, and one magnitude is not necessarily equal or unequal to another magnitude. Lines cannot be equal or unequal to surfaces, and a derived magnitude cannot be equal to an essential magnitude. In arithmetic, the A in $A \times B$ is a number of B's, or *so many* B's; in geometry, it is *so much* of B, where "*so much*" is more universal and applies to any *number* of a magnitude. If B is a magnitude, 3B may be a number, e.g., 3 polygons, or a magnitude, e.g., a line whose *length* is three feet; but $B\sqrt{2}$ and πB must be magnitudes and not numbers, for the coefficients are not numbers but *numbers* applicable only to magnitudes. This raises the problem of the meaning of such modern terms as "rational *number*" and "irrational *number*", assuming that applicability is retained.

The terms "rational" and "irrational" are contraries, and they arose in the field of magnitudes; for some magnitudes were considered as rational but others as irrational. A magnitude is neither rational nor irrational in itself but only in relation to another magnitude or magnitudes of the same kind; and this suggests the following definition. Similar magnitudes A and B are said to be rational if they can be measured (or numbered, if you wish) exactly by the same magnitude as a unit; otherwise they are irrational. Accordingly, such objects as 2/5 and $\sqrt{2}$ have universal applicability to magnitudes provided that some magnitude is posited as a unit of measurement to which they apply. Further, if any magnitude is posited as a unit, then 2/5 or $\sqrt{2}$ or π of that magnitude (or *so much* of that magnitude) always exists. The unit so posited may be of any size, e.g., one foot or one square mile or one cubic inch or one right angle or one radian. Thus it is evident that the symbol "$\sqrt{2}$", taken by itself, has no meaning apart from a magnitude, or else it belongs to a magnitude only; and the expression "$\sqrt{2}$ feet" signifies an object with possible existence. It is also evident that rationality and irrationality as contraries are not elements in arithmetic, for contraries belong to the same science (14a15-8, 427b5-6), whereas any two numbers are by definition measurable by a unit and do not need such contrariety. However, numbers admit a qualified contrariety with respect to measurement; for, if the unit of measurement is not a unit but a number, two numbers are measureable by that unit in some cases but are not in other cases.

The other laws given as axioms in mathematics need not be considered here; their discussion is similar to that of the commutative law of multiplication. The purpose of discussing this law here was to show that the law is meaningless or false or inapplicable to certain things under some presuppositions but meaningful and true and so applicable to those things under other presuppositions. Accordingly, if no presuppositions are made, the law may be true or may be false; and the same may be said of the theorems which follow by the use of it.

Usually, a mathematician guards himself against falsity or contradiction only within a limited field, and he so posits his axioms as to make consistency possible within that field, but with the absence of any further presuppositions or requirements. For example, he excludes division by zero to avoid inconsistency or falsity, and he need not give any reasons for it. But consistency is not guaranteed if the limits are removed. The two propositions "every A is a B" and "every B is a C" are consistent; but if we substitute man for A, stone for B, and mortal for C, along with the definitions of these things, there is inconsistency, and some conclusions are true but others are false. In general, a mathematician qua mathematician needs consistency within the limits of his subject, whether the axioms and hypotheses initially posited are true (which is Aristotle's position) or merely consistent (which is the position of many if not most modern mathematicians); but he needs truths also. For if those principles are merely consistent, he cannot proceed unless he uses principles as axioms to proceed; and some of these belong to logic and philosophy and must be true. For example, if he uses the *reductio ad absurdum* principle, he assumes both the principle of the excluded middle and that of contradiction, both of which are true and not just assumptions; for, if he posits their opposites, he destroys every science. To say that these two principles are linguistic or conventions or ways of speaking or of thinking is false; for, when one says that a thing cannot both have and not have an attribute at the same time in the same respect, he is dealing with matters of fact and not with language or convention. In general, if a mathematician refuses to posit any truth, he cannot contradict an opponent or deduce a theorem; for, in asserting an implication, e.g., that P implies Q, he must believe that this implication is always true, otherwise he can have no objection to the denial of that implication. Moreover, he must posit only true statements as principles, for if he accepts some assumptions as true but leaves other assumptions hanging, where does he draw the line?

Applicability is not the *reason* but a sign of the truth of assumptions. Assumptions are applicable because they are true; and they are true because of themselves and not because they are applicable. Further, true assumptions are better than false assumptions for two reasons, their truth and their applicability, but primarily because of their truth and secondarily because of their applicability. Since that which is better should be posited but that which is worse should be avoided, only true assumptions should be posited.

It is evident from the above remarks that if a mathematician wishes to lay down the foundations of mathematics, he must posit principles some of which lie outside of his field; and he must posit only true assumptions, whether mathematical or logical or psychological or philosophical, otherwise he is faced with difficulties. Further, he must posit elements which are appropriate to mathematics (71b19-25); otherwise his hypotheses concerning elements are left hanging, i.e., they may be true or false, depending upon further specification. Accordingly, the elements of mathematics or a branch of it must be proper to that science and not elements of any other science. For example, mathematicians must posit as elements such things as units and points and elementary magnitudes, lay down definitions and axioms and hypotheses, and proceed to theorems (76a31-6).

Returning to the subject of mathematics as a whole, Aristotle's subdivision differs from that of modern mathematicians. His first concern is with quantities taken universally, and the corresponding science, universal mathematics, investigates the properties of quantities qua quantities. Such properties are analogous and not univocal, for magnitudes and numbers, the two species of quantity, are not comparable; numbers cannot be added to or subtracted from magnitudes, e.g., the number 5 is neither equal nor unequal to a given line. Theorems in universal mathematics consider such things as equality, inequality, ratio, proportion, and whatever is common to all quantities by analogy. Universal mathematics presupposes and uses principles and theorems from higher sciences, e.g., from logic and philosophy (74a17-25, 1026a25-7, 1064b8-9).

The first subdivision of quantities is into numbers and magnitudes, and this subdivision differs from that of modern mathematics. The corresponding sciences are arithmetic (or theory of natural numbers) and geometry. Geometry includes almost all of what is nowadays known as analysis, algebra, and topology. The principles as elements of numbers are units only, and these are posited as indivisible and without position. But magnitudes have many kinds of elements, e.g., points and lines and surfaces and solids among essential magnitudes, and they have relative position and are, with some exceptions (as in the case of points), infinitely divisible, and some of these elements admit of differentiae, for lines may be straight or curved, and surfaces have similar differentiae. For these and other such reasons, geometry is far more complex than arithmetic (87a31-7). Arithmetic will be considered first, for units are prior in existence and in knowledge to points.

Units qua natures may differ, but the discussion of this difference belongs to philosophy and not to mathematics (1052a15-5a2). The arithmetician need only posit that units are treated qua indivisible, with the understanding that they are of the same kind if they are to be comparable with respect to equality, inequality, and other such attributes. As for the axioms of numbers, they are analogous to those of magnitudes (76a37-b2); but the discussion of analogy belongs to philosophy and not to mathematics, for the kinds of things which are taken as units may come under any category, e.g., they may be substances or colors or actions or relations, and the discussion of these taken as a whole or universally belongs to philosophy and not to any special science.

Since units are without position, there can be no principle of position in arithmetic. Consequently, such objects as —3 taken by themselves are impossible in this science, for the minus sign requires a principle of position, such as zero in the *number* system and the origin in analytic geometry. So since there is no principle of position in arithmetic, positive units or positive numbers, too, cannot come under arithmetic; for "positive" and "negative" are contrary terms, and these must come under the same science and require a principle of position. Hence units are just units, neither positive nor negative. Besides, since 3 men is an instance of 3, if —3 were a number, what would "minus 3 men" mean and how can minus 3 men exist? Such an object cannot even be imagined; so either it is impossible, or "minus 3 men" is meaningless.

Now subtraction of 3 men from 10 men or of 3 units from 10 units is possible, but such subtraction is from something — usually a greater number — and not from nothing. Hence "A=B" in arithmetic cannot signify an equality unless A is a number or a unit. The symbol "0=0", too, does not signify an equality, for 0 is not a unit or a number; nor is that symbol needed in arithmetic, for "A—B=0" may be replaced by the form "A=B", provided that A and B are units or numbers.

Similarly, in arithmetic there is no need for any symbol taken as a variable to signify a negative number. For example, let the form $x^2—z^2=my^2$ be solved for integral values, positive or negative. Can the phrase "positive or negative" be eliminated? If m is allowed to take on as values any integers, positive or negative, one is inclined to think that the solution of $x^2—z^2=my^2$ will be more general than if m is allowed to take on only positive integers (or better, only numbers in our terminology). If m is a positive integer, both sides become positive; and here Aristotle would merely replace "positive integer" by "number". But if m is a negative integer, both sides become negative; and the equality is either false or meaningless according to Aristotle's principles, for negative numbers cannot exist. But the form $x^2—z^2=my^2$ may be replaced by the form $z^2—x^2=ny^2$, where n is a positive integer (i.e., a number in our terminology), and the solutions of this form may then be sought. If x and y are taken to be negative, they are impossible objects. Evidently, Aristotle's formulation of the problem leads to practically the same solutions, for the solutions with a negative m are replaced by the corresponding solutions with a positive n. However, there is a difference; all solutions according to Aristotle's principles have existence, if "existence" is taken in the philosophical sense and not posited arbitrarily, but no solutions with a negative m or x or y can exist or be meaningful.

It appears, then, that the universality of the above problem and other such problems as stated nowadays is only apparent and leads to philosophical difficulties; for some values of the variables lead to truths but other values lead to falsities or absurdities. Now no one would deny that certain technical formulations of problems are convenient and save time; but such formulations need not be contrary to philosophical principles. Historically, the sciences since the Renaissance tended to take on a technical rather than a philosophical formulation, and for two reasons: less contact with philosophy and greater complexity of the problems; and getting results conveniently tended often to a change in the meaning of terms at the expense of philosophical distinctions· Thus the original terms "number", "unit", and "principal of number" lost their original meaning and use, and the terms "one" and "five" became mere instances of "number". It was then easy to say, on a superficial reading of ancient Greek mathematics and philosophy, that the Greeks had a limited concept of number and no concept of zero or of a negative number or of a fourth dimension, that the whole is not necessarily greater than the part, and to make other such distortions of Greek thought. Technical convenience in a science is certainly desirable, but it should be tied to a philosophical formulation if truth is to be maintained and difficulties are to be avoided. Accordingly, apparent generalities of a technical nature may be introduced for convenience in a science, but these must be so formulated that they do not disturb the philosophical formulation of that science. It is possible, for example, to introduce for convenience the term "negative integer" in the theory of numbers, provided that it is so defined that it does not signify a nonentity.

Since numbers are quantities, arithmetic may use principles and theorems from universal mathematics or other higher sciences. Universally, a subordinate science may use principles and theorems from a higher science.

The other part of mathematics is geometry, and its ultimate subjects are magnitudes, both essential and derived. Unlike a number, a magnitude is infinitely divisible; and its parts have order or relative position. As for indirect or accidental magnitudes, they are not the concern of geometry; for such magnitudes are or belong to nonmathematical subjects. We have shown in Comm. 8 of A, 7 in the *Posterior Analytics* that the cause of AB, where A is a mathematical attribute proved to belong to a mathematical subject B, lies in mathematics, and from this fact it follows that no mathematical trait can be a property of an indirect or accidental quantity. Geometry, then, is concerned only with the properties of (*a*) essential and derived magnitudes, and of (*b*) whatever belongs to these, considered generically or specifically or analogically. Most of modern mathematics (algebra, geometry, and analysis) would be included under geometry, for the ultimate subject of most of modern mathematics presupposes the continuum, which, according to Aristotle's philosophy, must be defined ultimately in terms of essential magnitude. The theory of numbers, of course, is excluded from geometry, but truths about numbers may be used in geometry.

Geometry itself, like mathematics, may be considered universally or specifically, for the properties of magnitudes qua magnitudes differ from the properties of specific magnitudes qua specific; and since the more universal is prior by nature to the less universal, the science of magnitudes, universally taken, precedes the science of specific magnitudes. Universal geometry, then, comes first, and this science investigates the properties (*a*) of essential and derived magnitudes qua magnitudes and (*b*) of geometrical attributes belonging to such magnitudes, and such properties belong to their subjects by analogy, for different kinds of magnitudes are not comparable. For example, two lines are either rational or irrational, but a line and a surface are neither rational nor irrational. This science is concerned universally with such things as inclusion, rationality and irrationality, limits, separation, and correspondence in magnitudes, for all of these are common to magnitudes by analogy. In general, if A is more universal than B and B is more universal than C, then the science of A precedes that of B, and the science of B precedes that of C. For example, the science of animals precedes the science of men; in algebra, the science of groups precedes the science of rings, and the science of rings precedes the science of fields; and, in modern geometry, the science of the affine group precedes the science of the Euclidean and other subgroups.

In the sciences, Aristotle considers another kind of priority also. If P is a part of W but not a genus of W, and if the definition of W includes P, then the science of P precedes the science of W. For example, a straight line is a part of a polygon (not a material part but a part as form, 1023b19-22) but not a genus of it, and it is included in the definition of a polygon. Accordingly, the science of straight lines precedes that of polygons. Similarly, the science of lines precedes that of surfaces, the science of surfaces precedes that of solids, and the science of functions of n variables precedes that of functions of $(n+1)$ variables. Of course, there is a science of functions of any number of variables, universally taken, and this is like the science of magnitudes, universally taken. But the science of, say, functions of one variable is to the science of functions of two variables as the science of lines is to the science of surfaces, and the science of functions of one variable is concerned only with attributes which are properties of such functions. As for the other attributes of functions of one variable, they belong to the science of functions of any number of variables, universally taken.

According to the priority just considered, the order of spatial geometries is as follows: line geometry, surface geometry, and solid geometry; for, with few exceptions, knowledge of surfaces and their properties presupposes knowledge of lines, and knowledge of solids and their properties presupposes knowledge of surfaces. Again, within each of these sciences there are other kinds of priorities. Thus knowledge of the elementary subjects in each science is prior to knowledge of subjects which are not elementary, e.g., knowledge of a straight line is prior to knowledge of a curved line and its properties. For example, in finding the *length* of a curved line, one must use the straight line, but not conversely; for one defines the *length* of a curved line as the limit of the sum $\Sigma(\Delta s)$, in which each Δs is a straight line, but one never uses a curved line — for curvature varies and so, unlike straightness, is not simple — to find the *length* of a straight line. The same applies to *areas* and the like. There is priority even within curved lines; for knowledge of the *curvature* of a noncircular curved line presupposes knowledge of the *curvature* of a circular line and hence of the circular line itself, as is evident from the definition of *curvature*, but the *curvature* of a circular line is known by its definition, which is a principle, and no *curvature* of any other curved line is prior in knowledge to that principle.

Concerning continuity in one or more dimensions, it depends on the meaning of the term "dimension"; for it is used in two senses: (*a*) spatial dimension, and (*b*) generic dimension, and the latter includes essential as well as derived magnitudes and is therefore more universal than spatial dimension. A straight line comes under spatial dimension, and the real *number* system is applicable to that line when a unit line as a principle of measurement is taken. But the real *number* system is applicable also to surfaces, volumes, weights, temperatures, and other magnitudes, each of which has its own unit as a principle of measurement. Accordingly, the magnitudes in the real *number* system are more general than spatial magnitudes; and let them be called "generic". The term "dimension" is used in other senses also. Spatial dimensions come under the spatial geometries already mentioned. The sciences of generic dimensions would be, according to Aristotelian principles, analogous to Aristotle's logic and first philosophy; for, just as the universality in these two sciences is one by analogy, in view of the analogous meaning of "being" and of "belonging" (or "predication"), so it is in the sciences of generic dimensions, in view of the analogous nature of the objects which come under each generic dimension. Let us now consider the sciences of dimension in sense (*b*), starting with the science of one-dimensional objects.

First, a relation is between things. For example, a father is a man in relation to something else, and a man is prior in knowledge to a father; for to know what a father is one must know what a man is, but the converse is not the case. Similarly, in the real *number* system, to know $+\pi$ and -3.5 and the like one must know π and 3.5, respectively, but the converse is not the case. Accordingly, there is a science of one-dimensional objects which are not relations but one-dimensional signless magnitudes, universally taken, and this science is one by analogy and is prior in knowledge to the science of what is known as the real *number* system, which includes positive and negative real *numbers*. Examples of such magnitudes in this science are lines, surfaces, volumes, angles, and all the rest of one-dimensional derived magnitudes, and each of them presupposes a unit of measurement; and one trait of two magnitudes, if they are of the same kind but different, is that one of them is greater than the other. Operations such as addition, subtraction, and the like are included in this science, provided that the resulting objects are magnitudes like those involved in the operations. For example, the result of (6.3 feet—2.1 feet) is 4.2 feet, and 4.2 feet is the *number* of a one-dimensional signless magnitude; and 6.3, 2.1, and 4.2 must apply univocally to magnitudes of the same kind but analogically to the various kinds. But (2.1—6.3) as an attribute of a signless magnitude is meaningless or does not exist, just as no such thing as minus two men exists. Concerning 0, it is not a magnitude but the privation or negation of it; for subtraction of one unit magnitude after another from a magnitude of 5 units finally ends in no magnitude, which here is a privation or negation of magnitude, and further subtraction is impossible.

As stated earlier, the units in arithmetic are studied qua indivisible; but an object such as π or $\sqrt{2}$ presupposes a unit which is infinitely divisible, and a one-dimensional magnitude is such an object. Now units may be either divisible or indivisible. If they are indivisible, or if they are divisible but are studied qua undivided (as when one considers apples as units but does not allow division of any such unit), then they are objects of arithmetic or objects to which arithmetic is applicable. But *numbers* such as π and $\sqrt{2}$ cannot exist or be exhibited unless a unit magnitude is posited. Accordingly, there is a science (*a*) which is concerned with generic *numbers*, taken as one-dimensional and signless, and (*b*) which must posit as a principle a unit which is continuous and one-dimensional. *Numbers* in this science include magnitudes which are exactly measurable; so 2 and 5 as magnitudes, too, belong to this science, as in the case of 2 square feet and 5 cubic yards. And the term "*so much*" rather than "*so many*" may be used in this science, as suggested earlier. Let this science be called "science A".

Signed numbers, such as $+2.3$ and $-\sqrt{2}$, presuppose signless *numbers*, as already stated, and they are not quantities but quantitative relations. So Russell and Whitehead are right in regarding such *numbers* as relations and distinguishing them from the corresponding signless numbers, which are 2.3 and $\sqrt{2}$. Further, signed *numbers* require a principle; for $+b$ and $-b$, qua relations, must be related to that principle, and $+b$ and $-b$ are contraries with respect to that principle. In a straight line, as a special case, this principle becomes a point which is called "zero point" or "origin", and $+b$ and $-b$ become, by convention, positions or vectors or directions to the right and left, respectively, of that principle; and, as positions, $+b$ and $-b$ are points relative to the principle, but as vectors, they are directional quantities relative to the principle. In another special case, the *numbers* $+b$ and $-b$ may be viewed as counterclockwise and clockwise angles relative to a straight line as a principle. There are other examples.

Let the science of signed *numbers* in one generic dimension be called "science B". Then science A is prior knowledge to science B. One may use the terms "zero", "vector", "position", and the like in science B, but these will have a more universal meaning; and this meaning will be one by analogy. In a similar manner, let science C be the science of signed *numbers* in two generic dimensions, science D be the science of signed *numbers* in three generic dimensions, and so on. If all these sciences have geometrical traits in common, then there is a science in geometry, say U, whose properties are common attributes but not properties in sciences A, B, C, and the rest.

In spatial plane geometry there are only two spatial dimensions and four spatial directions, and in spatial solid geometry there are only three spatial dimensions and six spatial directions (205b31-34, 208b12-19, 209a4-6), and such dimensions and directions are principles; but in a science which is an application of a science of more than three generic dimensions there are additional principles, as in the case of the science of functions of four variables. As for the corresponding proper principles of two species under the same generic science, some or all of them are one by analogy. Science B, for example, is applicable to a straight line with signed *numbers* and to the plane with signed angles; but one of the principles in the first application is the zero point, whereas the corresponding principle in the second application is an initial line in the plane with a point of rotation, and these two principles are one by analogy. Other corresponding principles are: distance and direction to the right in the first case, but angle and counterclockwise direction in the second case, respectively. Again, the generic function x^2 (or $y=x^2$) is applicable to the parabola in the plane and to the distance x^2 corresponding to each x along a straight line; but the parabola and the straight line have differences. The same applies to functions of two or more variables.

Now although corresponding geometric principles in specific sciences under a generic science in geometry are one by analogy, their analogical nature makes them also different in some respects, otherwise they would be univocal and not analogical; and it is from the differences between those principles that the properties in the specific sciences arise. In the above examples, the initial line and the counterclockwise direction differ, respectively, from the zero point and the direction to the right. So it is evident that some attributes in the science of signed *numbers* in a straight line are not attributes in the science of signed angles in the plane, and conversely. Further, since a specific science results by the addition of specific principles to those of the generic science, and since a property P in the specific science must require for its demonstration one or more of those specific principles as premises (otherwise P would be a property in the generic science), it follows, too, that (*a*) a property in the generic science is an attribute but not a property in the specific science and (*b*) a property in the specific science is neither a property nor an attribute in the generic science. For example, in the science of signed angles the line at a distance of $2n\pi$ (where n is an integer) from the initial line coincides with the initial line, but in the science of signed *numbers* in a straight line no point at a distance from the origin coincides with the origin.

The same is true even if the common principles of two specific sciences which come under a more universal science are one univocally and not analogically. For example, the concurrence of the medians is a property of a triangle but an attribute and not a property of a right triangle; and the property $a^2+b^2=c^2$ concerning the sides of a right triangle is neither a property nor an attribute of a triangle, for it must use in the premises of its demonstration the triangle's right angle, which is a principle in the science of right triangles but not in the science of triangles. Universally, then, no property of a more universal science is a property of a less universal science, and conversely.

It follows from the above analysis that science B is necessary but not sufficient for the science of signed *numbers* in a straight line, and consequently that the latter science is not reducible to science B. Further, the science of signed *numbers* in a straight line is a part of geometry and hence of mathematics, and any property in it must use for its cause principles which are not in science B; hence any definition which excludes this science from being a part of mathematics is inadequate to that extent. The same remarks apply to any other science coming under a generic science in geometry. These conclusions are further proof that the position of those who limit mathematical investigations to the so-called "formal properties" is inadequate, badly formulated, and mistaken.

It is evident from what has been said that *numbers*, functions, and all other objects in which continuity enters, whether explicitly or implicitly, are objects treated in the various generic and specific geometrical sciences. Of those objects, spatial magnitudes and the attributes of these qua such magnitudes are essential but the rest are derived; and of spatial magnitudes, solids are the ultimate subjects of all other essential and derived magnitudes and their attributes; for lines and surfaces cannot exist without solids, and derived magnitudes and their attributes are posterior in existence to essential magnitudes and hence to solids. Of the traits of magnitudes, whether spatial or generic, some are quantities but others are not; and those which are not quantities are attributes of quantities. Thus function, incommensurability, isomorphism, congruence, mapping, and other such traits which appear in analysis, geometry, and abstract algebra will have to include, in Aristotle's systematic formulation of a science, continuity in their philosophical definitions, whether directly or indirectly (1030b16-26, 1045b29-32). Two examples may be taken. Incommensurability is not a quantity; and it is defined as impossibility of two similar one-dimensional signless magnitudes of being numbered by the same magnitude as a unit. A one-to-one dimensional function may be defined as a one-to-one relation, in accordance with a geometrical relation (or equality) of two one-dimensional magnitudes or of two continuous sets of attributes of the same magnitude. Specifically, the side and volume as signless *numbers* of a cube are so related, and so are the continuous sets of abscissas and ordinates as signed *numbers* of the line whose equation is $y = x^3$. A function, universally taken, will have to be defined in a more general way.

Historically, Greek mathematicians were concerned mostly with demonstrations of attributes of essential and specific quantities, for one would hardly expect scientists at the early stages of science to be dealing with the most abstract subjects. Plato and Aristotle, on the other hand, were quite familiar with the degrees of universality. Aristotle mentions the failure of early Greek mathematicians to treat certain mathematical objects more universally (74a17-25), and, in discussing causation in a demonstration, he includes the highly abstract demonstrations through causes each of which is one by analogy (99a1-21). Within geometry, such causes are necessary in the generic sciences. But Euclid, as I have shown elsewhere (*Methodological Superiority of Aristotle over Euclid*, "Philosophy of Science", April 1958), though later than Aristotle, was not familiar with Aristotle's formulation of causation in a science; for his *Elements* dealt mainly with essential quantities and showed a lack of analogical causation and scientific organization. It is evident, then, that Aristotle regards causation by analogy as a part of scientific method; and, among scientists, such causation is used by mathematicians most of all.

Some remarks concerning relations, axioms, definitions, and language deserve attention.

Relations have properties and, like demonstrations, exist in every science; hence just as there is a science of demonstration, so there should be a science of relations. If we attend to Aristotle's general principles, it appears that there should be a science of relations; for, he says, "of every genus of things there is... one science" (87a38-39, 1003b19-20), and relation is a category and hence a genus (83b13-17). But in his list of the sciences no mention of a science of relations is made. Perhaps this is because he regarded a relation as a sort of offshoot of being (1096a20-22) and as being least of all a nature or substance and also as being posterior in existence and in knowledge to the other categories (1088a22-b1). A science of relations is a recent phenomenon, and it has been developed by mathematicians and mathematical philosophers whose technique and sense for accuracy is most pronounced. But it cannot be a part of mathematics, as stated earlier, for relations exist in every science. It follows, then, that it is an independent science and is applicable to mathematics as logic is applicable to all sciences and as mathematics is applicable to physics, chemistry, economics, and other sciences. In demonstrating mathematical properties, of course, it makes no difference whether the science of relations appears as a part of mathematics or as being applicable to mathematics; and the fact that short symbols are used for relations, as it is for mathematical objects, should not mislead one to regard the study of relations as belonging to mathematics. In short, relations are between things which may come under any category; so the study of relations between mathematical objects would not be the most universal and hence cannot lead to properties, which follow only through the cause.

Many definitions in mathematics are given in terms of axioms. For example, equivalence (or equality) is defined as a relation between things subject to the axioms of reflexiveness, symmetry, and transitivity. Now the modern use of the term "axiom" is much wider than that of Aristotle, for it includes even what Aristotle regards as definitions. For example, Aristotle would define a (plane) triangle as a three-sided plane figure, and the terms "three", "sided", "plane", and "figure" must be taken in that order for certain philosophical reasons (1038a9-16). A mathematician may replace this definition by the statement that a triangle is defined as an object x subject to the axioms: (*a*) x is plane, (*b*) x is a figure, (*c*) x has three and only three sides. Thus Aristotle's distinction between an axiom and a definition does not appear in modern mathematics, and as far as getting properties is concerned, perhaps it makes no difference. But a number of problems may be raised with respect to the nature of axioms, definitions, hypotheses, and other kinds of principles. Such problems, however, are philosophical, and their solution hardly affects the various definitions of mathematics and the problems under discussion in this treatise; so we leave them aside here.

A problem which may be mentioned is whether certain principles posited nowadays as axioms are about mathematical objects or about their symbols; for, as shown earlier, there is no one-to-one correspondence between the attributes of things and those of their symbols. Thus if the axiom of symmetry (i.e., the axiom that if $x=y$, then $y=x$) is intended for the symbols, which in "$x=y$" and "$y=x$" differ in position, it is false for certain objects, for these may be numbers, and numbers may be equal but have no position; but if it is intended for those objects, then a discussion of the meaning of the axiom is needed to show that the axiom is true for the objects intended. The mere positing of the axiom as defining (along with other axioms) equality is not sufficient for its applicability to numbers and other mathematical objects; and if it is posited as applicable only to the objects for which it is true, without identifying those objects, we are left in the dark as to what those objects are, and both the truth and the applicability of the axiom are left out. In practice, of course, the part of the axiom which refers to objects without position as having position may not affect the conclusions, and in such cases the theorems which follow are true and applicable. Logically, however, the theorems are true in spite of the axiom, for true conclusions may follow from false premises also, and the truth of such conclusions is accidental and does not follow from principles which are causes. But *knowledge* of a theorem must be through the cause.

Again, failure to grasp the meaning of terms as used by past thinkers leads to unawareness of their contribution and to unfair criticism and evaluation. The criticism of past philosophy by the Logical Positivists is a classical example, and so is the modern evaluation of the philosophy of ancient Greek mathematics. Did the Greeks have the concept of zero? Certainly. But must one use one word or the word "zero" or the symbol "0" or the like in order to have that concept? As indicated earlier, Aristotle's equivalents of the concept of zero are "privation of quantity" in the case of signless quantities and "principle of position" in the case of signed quantities. Did Aristotle have the concept of a negative number? From his principles it follows that such a number, as a real *number*, is a *number* as a position relative to the principle of position and on the side toward one of the two contraries. The same applies to signless *numbers*. Is the whole ever equal to the part? If "whole" and "part" and "equal" are taken in the sense in which the Greeks used them, the answer is negative; and if they are taken in a different and wider sense, we have the following alternatives. (*a*) The Greeks were still right, for they did not use those terms in the wider sense. (*b*) If equality is defined by a one-to-one correspondence, by which a part of a line appears nowadays to have as many points as the whole line, one is faced with Aristotle's arguments which deny the hypothesis that a line consists only of points as its matter and which assert that distance is a principle which cannot follow from such a definition of equality. The concept of an open set, which is thought to be a recent discovery, was first introduced by Aristotle in his *Physics* (236a7-b18), though a more appropriate term for him would be "open region", for he denies a continuum as consisting of indivisibles as matter. Did Aristotle have the concept of modern physics? He certainly did, if one considers the manner in which he subdivides subjects and scientific problems. All non-mathematical problems concerning modern physical subjects, generically taken, come under his so-called "physics", whose subject, generically taken, is wider than that of modern physics. All problems which are mathematical applications to modern physical subjects come under his so-called "mechanics"; for he says that mechanics is concerned with the mathematical treatment of physical subjects (847a24-28). The problem of inertia is considered, but in different terminology. It is discussed qualitatively in 266b28-267a5, where Aristotle says that the object moved becomes a mover, which amounts to saying that the object moved gains momentum from a mover; and if resistance and direction and the mathematical attributes of movers and objects moved, which are discussed in other parts of his works, are taken into consideration, we arrive at something which amounts or is close to Newton's law of inertia. Other examples may be mentioned.

Conclusion.

According to Aristotle, the unity of a theoretical science lies in a genus as a subject, and in mathematics that genus is quantity. Ideally, the mathematician should investigate theorems, i.e., necessarily true conclusions through their causes, which are necessarily true principles. Theorems are propositions of maximum generality, state necessary and sufficient conditions, and signify properties as belonging to subjects. The subject of a theorem may be a quantity or an attribute of a quantity; and, as a subject, a quantity or an attribute of it may be taken at any level of universality, i.e., it may be taken analogically, or generically, or specifically, and there are many levels in each case. Since the more universal belongs to the less universal, the more abstract parts of mathematics are prior in *knowledge* to the less abstract, and their investigation, considered scientifically, should precede that of the less abstract. Further, since mathematics is not the most abstract science, it must use some principles from the more abstract sciences, e.g., from philosophy and logic; for it must lay down definitions according to the principles of a definition, and definitions come under philosophy, and it must demonstrate theorems in accordance with the truths of logic.

The limitation of mathematics to quantity as the unifying subject is no bar to the indefinite growth of this science; for (*a*) growth within a genus occurs through new concepts, new definitions, new axioms, and new theorems, and (*b*) the various subjects whose properties are to be investigated may be kinds of quantities or of attributes of these quantities, and such subjects — especially the attributes — are very numerous and perhaps infinite, and so are the corresponding parts of mathematics.

Modern mathematicians say that they are not interested in the truth or falsity of their postulates; actually, however, almost all of their postulates are philosophically true and have possible applications. Aristotle would (*a*) object to the manner in which postulates are nowadays distributed, for he would regard some modern mathematical postulates as belonging to other sciences, (*b*) attribute a certain amount of equivocation and indefiniteness to modern mathematical terminology, and (*c*) regard few mathematical postulates as false or meaningless.

Quantitatively taken, however, more than ninety percent of modern research in mathematics comes under Aristotle's definition of mathematics.

Η ΑΡΙΣΤΟΤΕΛΙΚΗ ΘΕΩΡΙΑ ΤΩΝ ΜΑΘΗΜΑΤΙΚΩΝ ΩΣ ΕΠΙΣΤΗΜΗΣ ΠΟΣΟΤΗΤΩΝ

Περίληψη.

Δὲν εἶναι σπάνιο μιὰ γενεὰ νὰ θέλη νὰ ἐξυμνῆ τὰ δικά της ἐπιτεύγματα καὶ νὰ ὑποτιμᾶ ἢ ἀκόμη νὰ ἀγνοῆ τὰ ἐπιτεύγματα τῶν γενεῶν ποὺ πέρασαν. Ἐξαίρεση σ' αὐτὸ δὲν κάνουν οὔτε οἱ γενεὲς τοῦ αἰῶνα μας. Ἂν τώρα ἡ τάση αὐτὴ ὀφείλεται σὲ ψυχολογικὴ ἀδυναμία ἢ γίνεται γιὰ νὰ κάνη ἐντύπωση στοὺς πολλούς, ἴσως τὸ κακὸ νὰ μὴν εἶναι τόσο μεγάλο· ἂν ὅμως υἱοθετῆται ἀπὸ ἐπιστήμονα ἢ ἀπὸ φιλόσοφο, ποὺ ἀσχολεῖται μὲ τὴ φιλοσοφία τῆς Ἐπιστήμης ἀλλὰ δὲν μελετᾶ καὶ τὴν ἱστορία της, τότε ἡ τάση αὐτὴ συντελεῖ στὴν παραπλάνηση τοῦ ἐπιστημονικοῦ κοινοῦ.

Κάτι τέτοιο συνέβη καὶ μὲ τὴν ἱστορία τοῦ ὁρισμοῦ ὡς καὶ μὲ αὐτὴ τὴν Φιλοσοφία τῶν Μαθηματικῶν κατὰ τὴν διάρκεια τουλάχιστο τῆς τελευταίας ἑκατονταετίας. Διαπρεπεῖς μαθηματικοὶ καὶ φιλόσοφοι, ποὺ ἐπεχείρησαν νὰ προτείνουν ὁρισμὸ γιὰ τὰ Μαθηματικά, ἢ ἔδωσαν ἁπλοϊκὴ ἀναφορὰ τῆς συμβολῆς τοῦ παρελθόντος ἢ ἀγνόησαν τελείως τὴν συμβολὴ αὐτή. Ἔτσι, ἐπιδεικνύοντες μιὰ τέτοια ἀντιεπιστημονικὴ συμπεριφορά, ὄχι μόνον δὲν συνέβαλαν σὲ ὁποιαδήποτε πρόοδο ἀλλὰ καὶ ἐπανέλαβαν ἱστορικὰ λάθη.

Ἡ ἀναγωγὴ κατὰ τὸν Russell τῶν Μαθηματικῶν στὴ Λογικὴ εἶναι κατὰ βάση ἀνάλογη μὲ τὴν πυθαγόρεια καὶ πλατωνικὴ ἀναγωγὴ ὅλων τῶν πραγμάτων σὲ ἀριθμοὺς καὶ σὲ ἀρχὲς τῶν ἀριθμῶν. Ἀκόμη, ἡ κατὰ τὸν Russell ἀναγωγὴ τῶν προτάσεων ἀφετηρίας (ἢ ἡγουμένων προτάσεων) σὲ ὀλιγάριθμο πλῆθος ἀρχικῶν προτάσεων εἶναι ἀνάλογη μὲ τὴν πλατωνικὴ ἀναγωγὴ ὅλων τῶν ἐπιστημῶν στὴν ὕψιστη ἐπιστήμη, τὴν Διαλεκτική. Ὡστόσο, τέτοιες ἀναγωγὲς εἶναι λογικὰ ἀδύνατες, ὅπως ἠμπορεῖ κανεὶς νὰ δείξη. Ἐξ ἄλλου, ὁ ὁρισμὸς τῶν Μαθηματικῶν ἀπὸ τὴν Ἐνορατικὴ Σχολὴ παρέχει

παρόμοια ἀναγωγὴ γιὰ τὰ μαθηματικὰ ἀντικείμενα σὲ ἀριθμούς, τῶν ὁποίων ἔχομε ἀντίληψη μὲ μιὰ βασικὴ ἐνόραση. Τέτοια ὅμως κατὰ γράμμα ἀναγωγὴ περιορίζει τὰ μαθηματικὰ ἀντικείμενα στοὺς φυσικοὺς ἀριθμούς, καὶ τοῦτο γιατὶ ἡ συνέχεια, ὡς καὶ μερικὰ ἄλλα μαθηματικὰ ἀντικείμενα, δὲν ἠμποροῦν νὰ ἀναχθοῦν σὲ φυσικοὺς ἀριθμούς.

Οἱ Λογικοθετικιστὲς τοῦ αἰῶνα μας ξεκίνησαν μὲ τὴν παραδοχὴ ὅτι ἡ φιλοσοφία πρὶν ἀπὸ τὴν τελευταία ἑκατονταετία «ἐστερεῖτο ἐξ ὁλοκλήρου νοήματος». Προφανῶς οἱ ἐρευνητὲς αὐτοί, ὡς καὶ ἄλλοι νεώτεροι, ποὺ προτείνουν νέους ὁρισμοὺς γιὰ τὰ Μαθηματικά, δὲν ἐμελέτησαν τὴν Ἱστορία τῆς Ἐπιστήμης αὐτῆς. Μάλιστα, ἐκεῖνοι ἀπὸ αὐτούς, ποὺ θεώρησαν ὡς πολὺ περιωρισμένο τὸν ὁρισμὸ ὅτι «τὰ Μαθηματικὰ εἶναι ἡ ἐπιστήμη τῆς ποσότητος», δὲν ἐγνώριζαν οὔτε τὴν σημασία τοῦ ὅρου ποσότης, ὅπως ἐχρησιμοποιεῖτο ἀπὸ τὸν Ἀριστοτέλη, οὔτε τὶς ἀρχὲς σύμφωνα μὲ τὶς ὁποῖες διετυπώθη ὁ ὁρισμὸς αὐτός.

Σκοπὸς τῆς παρούσης πραγματείας εἶναι ἡ παρουσίαση μὲ ἐπιστημονικὸ τρόπο τοῦ ὁρισμοῦ τῶν Μαθηματικῶν ἀπὸ τὸν Ἀριστοτέλη, ὡς καὶ τῶν ὁρισμῶν ποὺ δόθηκαν ἀπὸ νεώτερους ἐρευνητές, ἡ συσχέτιση τῶν ὁρισμῶν ἐκείνων μὲ τὰ μαθηματικὰ ἀντικείμενα καὶ τὴν ἔρευνα, καθὼς καὶ ἡ κατάδειξη τῆς ὑπεροχῆς ποὺ ἔχει ὁ ἀριστοτελικὸς ὁρισμὸς σὲ σχέση μὲ νεώτερους ὁρισμούς.

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