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THE SPIDER IN THE SPHERE. EUDOXUS' *ARACHNE**

Is there another scientist in the whole history of science whose craftsmanship has been as deep-plo yetughing terse, as lasting yet prolific, as Eudoxus of Cnidus's contribution to the development of mathematical analysis? Aristotle's influence has been lasting, too, but since the Middle Ages mostly retarding, and Plato's mostly indirect. But as regards analysis,

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* This is a revised and generalized version of a lecture delivered at the «Conference on the History of Measurement. The Role of Measurement Standards in Human Civilization» held in April 27-30, 1976 in Budapest, Hungary. The general theme of the Conference restricted the lecture to the early history of angular measurement, and the philosophical implications of the measurement techniques described were not discussed. — References to formulae are made in round brackets, references to literature in square brackets:

[1] Kasanen E., «Appendix. An Algebraic View» in [9].
[2] Mattila J., «Appendix» in [11].
[3] Mattila J., *On Some Mathematical Properties in Plato's «Great Harmonia». A Study on Harmonious Points and their Applications,* in *Analysis, Harmony and Synthesis in Ancient Thought,* Ann. Univ. Ouluensis 1976 (forthcoming).
[4] Maula E., *Plato's Agalma of the Eternal Gods,* «Ajatus. Yearbook of the Philosophical Society of Finland» 31 (1969).
[5] Maula E., *Plato's «Mirror of Soul» in the Timaeus,* ibid. 32 (1970).
[6] Maula E., *Plato's «Cosmic Computer»* (*Tim. 35a - 39e*), ibid, 32 (1970).
[7] Maula E., *Ancient Shadows and Hours,* «Ann. Univ. Turkuensis», Ser. B, Tom. 126, Turku 1973.
[8] Maula E., *Studies in Eudoxus' Homocentric Spheres,* Commentationes Humanarum Litterarum, Vol. 50, Helsinki (Societas Scientiarum Fennica) 1974.
[9] Maula E., *The Constants of Nature. A Study in the Early History of Natural Law,* «Φιλοσοφία», Vol. 4 (1974), 210-246.
[10] Maula E., *The Elements of Analysis,* Proceedings No. 3 of the XIV International Congress of the History of Science in 1974 in Tokyo and Kyoto, Japan.
[11] Maula E., *Points of Contact Between Physics and Philosophy in their Early History. The Geometrical Spider,* in *On the Foudations of Physics;* Report Series in Theoretical Physics, University of Turku, Finland, 1975.
[12] Maula E., *The First Beats of a Dynamic World-View,* mimeographed paper, Dept. of History, University of Oulu 1975.
[13] Maula E., *The Return to Cnidus,* mimeographed paper, Dept. of History, University of Oulu 1976.

Weierstrass and Dedekind, more than two thousand years later, could set out directly from Eudoxus's results, preserved in Euclid's *Elementa*. Indeed, Eudoxus «was a man of science if there ever was one», as Sir Thomas Heath puts it.

What are the methodological insights behind these unique achievements? There is no extant example of Eudoxus's methods of computation, no example of his actual use of analysis, although we know some of his results. Nor do we know more about his methods of proof than that he perfected the method of exhaustion and used indirect proof. His methods of invention are utterly unknown. As for his instruments, we know the name of one of them only: the *arachne*.

Thus it seems immensely worthwhile to attempt a reconstruction of Eudoxus's other great achievements, his theories of mathematical geography and above all his cosmological system, in order to obtain new information about his methodological insights as well. True, Eudoxus's astronomy is a most rewarding object of research even in itself because, to quote Heath again, «no more ingenious and attractive hypothesis than that of Eudoxus's system of concentric spheres has ever been put forward to account for the apparent motions of the sun, moon and planets». For us, however, our previous reconstructions of Eudoxus's method of determining the geographical latitude [7], his astronomical methods and his methods of analysis and synthesis [8, 9, 10] have served as starting-points for an even more inclusive reconstruction of Eudoxus's scientific portrait and dynamic world-view, containing the nucleus of a dialectics of nature [11, 12].

As an important mile-stone on this highway, we now submit a reconstruction of Eudoxus's instrument, the *arachne* (Spider, mentioned by Vitruvius), in which his mathematical methods, observational techniques, and heuristic insights assume a concrete manifestation. Being basically a model of the geocentric system the *arachne* stands on top of a comprehensive yet strictly coherent theoretical foundation, which admits also of its other usages as an instrument for angular measurement, as a mathematical instrument for instance for the extraction of square and cubic roots, and as an aid in mathematical invention. Yet there is no extant description of it.

On the Sources and Choices.

Our strategy has been to approach the intertwined problems of Eudoxus's observational and computing techniques, and their heuristic ideas, by exploring his astronomical and geographical theories. Although our reconstructions in these directions have been presented earlier, it may be worthwhile to dwell on their general character even here. For, as is known, there is the now canonical interpretation of the theory of the homocentric spheres by Schiaparelli, whom most historians of science still follow, and as regards Eudoxus's method for the determination of geographical latitude, the later Hellenistic method is usually taken to account for his few numerical paraphrases for the latitude, even though the resulting inaccuracies in geographical location are simply unbelievable and incompatible with biographical tradition. However, Schiaparelli's interpretation is essentially q u a l i t a t i v e in the sense that he makes conjectures about the unknown Eudoxan parameter values in order to ascertain a fair correspondence between the general outline of Eudoxus's system and unchanging, observable facts. But he does not attempt a deduction from the known Eudoxan parameter values to observations, which might turn out to be even grossly mistaken. On the contrary, Schiaparelli and others in his wake, like Heath and Dreyer, are willing to jettison even Aristotle's presumably well-informed testimony on the details for the sake of their own interpretation. In this process, Eudoxus's computing techniques must be represented by qualitative geometrical constructions. It is as well to remember that the choice between a «geometrical» and a «physical» interpretation is a moot point (see e.g. Larry Wright, *The Astronomy of Eudoxus: Geometry or Physics?*, «Stud. Hist. Phil. Sci.» 4 [1973] No. 2), but we cannot consider such qualitative interpretations satisfactory. For we know for sure that Eudoxus used at least some exact parameter values, for instance that he gave the planetary periods in days and years. What we offer instead is a q u a n t i t a t i v e interpretation.

Our program for a quantitative interpretation is this: we begin from the relatively few known numerical parameter values and construct one and only one method of computation which yields numerical values for a l l other Eudoxan parameters described so far only qualitatively. Among these computed values there are some that represent instrumental observations. Yet one cannot know in advance, as Schiaparelli thought, whether these observations were fairly accurate. On the contrary, we know that some were idealized and some even utterly fictitious, for instance those pertaining to the third solar motion postulated by Eudoxus. Hence modern observations (combined with modern mathematics and astronomical tables giving e.g. the real *loxotes* in Eudoxus's day) will not help us in the way Schiaparelli believed. For a genuine reconstruction of Eudoxus's method of computation and observation must be capacble of reproducing even his idiosyncracies, and not only the unproblematic, well-known general features of his system.

But even though this could be achieved, the reconstructed method might do no more than illustrate a possible way of invention —presuming that

Eudoxus, too, started from just those parameter values that have been preserved. Pains must be taken to demonstrate that the reconstruction remains within the general framework of ancient mathematics, just as the observational instruments must lie within the scope of ancient technology. Moreover, one must outline a plausible route of actual computations from the observations to the resulting parameter values.

Now it is not likely for instance that Heath would have overlooked any relevant information in the ancient sources directly pertinent to Eudoxus. Hence additional, even though perhaps indirect, information must be sought from related sources. To our delight we have discovered that Plato, Eudoxus's contemporary and associate, gives us two invaluable clues in exact numerical terms in the construction of the world-soul in the *Timaeus*. One points towards the methods of numerical analysis in Pythagorean mathematics, the other towards an exact value for the obliquity of the ecliptic by these methods. We need not overstress the alleged, though likely, collaboration of Eudoxus and Plato. Plato's frame of reference may have been Eudoxus's system, but, alternatively, Plato's world-soul may have given an impetus to Eudoxus's cosmology. It is enough to say that this additional information suffices for our reconstruction of Eudoxus's methods of computation in astronomy and geography. For more than one hundred main parameters, including Eudoxus's fictitious parameters, can be computed starting from the known planetary periods [9]. It may be added that as far as real observational parameters are concerned, Schiaparelli's conjectures eventually turn out to be quite good (in the region of 2^0 - 3^0 from the computed ones).

Although the possibility of a mere coincidence in solving such a great number of exact parameter values is negligible, we suspect that many modern commentators may still oppose our use of additional information draw from the *Timaeus*. We are, however, quite convinced that Plato's *«great harmonia»*, described in full earnest and most carefully worded, can be tapped for new astronomical information then current in the Academy. For it can be shown that in general, when any two line segments partly overlap so that both are equally divisible by the overlapping part, their endpoints create a set of harmonious points the ratio of division being either 2 : 1 or 3 : 1 (depending on the point of view adopted). And Plato's *«great harmonia»* is constructed starting from just these basic «double and triple intervals». This highly interesting result is communicated in [2, 3] and constitutes a major step in our argument for the correctness of our Eudoxan reconstructions.

We interpret this result in terms of a discovery of Plato's rationale of the *«great harmonia»*, and shall include the use of harmonious points in the arsenal of ancient mathematical tools legitimately used in our reconstructions. In fact we make use of harmonious points both in the scales of the *arachne* and in the outline of Eudoxus's practical computations.

Now it must be noted that harmonious points in the ratios of division mentioned (2 : 1 and 3 : 1) are obtained as soon as Plato's basic double and triple intervals are interpreted in terms of line segments added to one another ($1 + 2 + 4 + 8$ and $3 + 9 + 27$), and their harmonic and arithmetical means, together with the additional intervals of 9 : 8 (all of them clearly stated by Plato), are likewise interpreted and inserted between the basic intervals. And according to Plato, they are so interpreted and inserted [8].

Here, a different arrangement of Plato's basic intervals in one row (e.g. 1, 2, 3, 4, 9, 8, 27) is advocated by Taylor and Cornford, who consequently see little or no mathematical significance in Plato's «*great harmonia*». But their arrangement is simply in contradiction to the text, for Plato explicitly says that when the insertions are executed, the intervals of 3 : 2, 4 : 3, and 9 : 8 are created, whereas the insertions into the series of the basic intervals arranged in one row produce an extra, unwarranted interval [8, p. 34]. Moreover, the better arrangement in two rows, in the form of a «lambda -like diagram» (Crantor's diagram) suggests a value for the obliquity of the ecliptic (ε), which is a central feature in Plato's astronomical model, in terms of a Pythagorean, right-angled triangle. For an acute angle (ε) in the triangle with the hypotenuse of $39 = 3 + 9 + 27$, and the shorter side of $15 = 1 + 2 + 4 + 8$ units of (relative) length, is obtainable from $\tan \varepsilon = 5{:}12$ or $\sin \varepsilon = 5{:}13$. And the same result is also obtained if one begins with two integers $p = 3$, $q = 2$ generating Pythagorean triples, i. e. with the same integers that figure in the formation of harmonious points by overlapping line segments, for in this case $C = p^2 + p^2 = 13$, $A = p^2 - q^2 = 5$, and $B = 2pq = 12$, where $A^2 + B^2 = C^2$ (and $p > q$; p, q are relatively prime and not both odd) [8, 9].

In this way we have deduced the o b l i q u i t y o f t h e e c l i p t i c in Plato's cosmological system in the *Timaeus,* and since this must have been a central parameter value generally known in the Academy, we further assume that even Eudoxus used precisely this value. Our final results make it clear that the assumption is correct.

To the possible objection that $\varepsilon_{plato} \approx 22^{0}37'$ obtained in this manner is less than the real value in Plato's and Eudoxus's time $\varepsilon_{real} \approx 23^{0}44'$ (see e.g. D. R. Dicks, *Early Greek Astronomy to Aristotle,* Ithaka / New York 1970, p. 154, n. 240), we answer that so it indeed should be in Eudoxus's system, for Eudoxus is known to have credited the Sun with a fictitious deviation from his ecliptic. This implies that the *loxotes* adopted by Eudoxus was somewhat too small as compared with actual observation. But even the fictitious additional deviation can be obtained by exactly the same method [9]. That is to say, we have here a deliberate «theoretical error», committed for methodological reasons.

There are ,of course, other interpretations of details based on the additional information we have drawn from Plato, and other smaller discoveries about the numerical parameter values of Eudoxus (including a computational connection between the synodic periods [9, p. 220]), but this may suffice to illustrate our use of the sources and some decisions made in their interpretation. More details will emerge from our description of the different usages of the *arachne* (below).

Eudoxus's Method of Celestial Computations.

In the theory of the homocentric spheres, the basic problem of explanation concerns the computations needed in the combination of two spherical motions. If these motions are characterized by angular velocities, the combination assumes the following form:

$$(1)\ \pm\ \omega_{I}^{ind\ (W\ or\ E)} \pm \omega_{II}^{ind\ (W\ or\ E)} = \pm\ \omega_{II}^{comb\ (W\ or\ E)}$$

Here the indices (I, II) refer to two *spheres*, the indices (ind, comb) to the individual and resulting motions of these spheres, and (W, E) to the directions of rotations. These are among the Eudoxan parameters qualitatively described in the tradition. Some of them can easily be given even an exact value. In fact the problems are not difficult as far as Eudoxus's first and second spheres for the seven planets are concerned. The real problems appear in the combination of his second and third, or second and fourth spheres. Hence we may take these as our examples. Systematical study of all alternatives is undertaken in [8]. Since $\omega = 1/T$, or the inverse of rotation time, and the directions of the second and third planetary motions are (partly) known from the tradition, their rotation times are obtained from:

$$(2)\ T_{III}^{comb} = T_{II}^{comb}\, T_{III}^{ind} : (T_{II}^{comb} \pm T_{III}^{ind})$$

It is shown in [9] that if an auxiliary parameter *n* is introduced in accordance with *Elementa* V. 15, and (2) is considered as a generalized proportion

$$(3)\ T_{II}^{comb}\, T_{III}^{ind} : (T_{II}^{cemb} \pm T_{III}^{ind}) = nT_{III}^{comb} : n,$$

and means are discovered for mastering (3), all Eudoxan parameters characterizing the theory of the homocentric spheres can be given exact numerical values.

The solutions have the following form in terms of the sides a, b, c of a general right-angled triangle, or in terms of two integers p,q generating a similar Pythagorean triangle in the usual way ($p > q$, p and q are relatively prime and *both* odd):

(4) for $xy:(x+y) = nT^{comb}:n$

$$\begin{cases} n=x+y=c+b \rightarrow (p+q)^2 \\ x=(a+b+c):2 \rightarrow p\,(p+q) \\ y=(b+c-a):2 \rightarrow q\,(p+q) \\ x:y=(a+c):b=p:q \end{cases}$$

(5) for $xy:(x-y) = nT^{comb}:n$

$$\begin{cases} n=x-y=c-b \rightarrow (p-q)^2 \\ x=(a-b+c):2 \rightarrow p\,(p-q) \\ y=(a+b-c):2 \rightarrow q\,(p-q) \\ x:y=b:(c-a)=p:q \end{cases}$$

It will be seen that the auxiliary parameter n disappears in the solutions.

How these solutions are obtained by using the methods of analysis and synthesis (this being a direct contribution to a better understanding of Eudoxus's ideas of analysis) is explained in detail in [9,11]. The solutions are perfectly satisfactory as regards ancient mathematical tools[1], and one can hardly imagine simpler solutions than ours. Moreover, the solution $x:y=p:q$ alone is really needed, and in [11] it is shown that these solutions, which are invariable with respect to the sign in (3), have a simple geometrical interpretation. For if in a Pythagorean right-angled triangle $\tan\alpha = 2pq:(p^2\text{-}q^2)$, then $\tan(\alpha/2) = q:p$ by *Elementa* VI.3. This fact we have made use of in the construction of the measuring unit of the *arachne*.

The most-far reaching idea here is the preference of ratios of relatively prime numbers for ratios of composite numbers. We can see this idea develop from integers in the ratio known as *superparticularis*, $(n+1):n$, which are met in musical consonances, and again in the «oblong numbers», $(n+1)n = 2+4+\ldots+2n$. But already in Archytas' (who was Eudoxus teacher) preserved proof that there is no (integer) number which is a (geometric) mean between $(n+1)$, n we meet also the relatively prime numbers. A definition is seen in the *Elementa* vii, Def. 12.

Despite the geometrical interpretation of Eudoxus's solution $x:y = p:q$ and its easy instrumental realization, we are still speaking about the possible way of Eudoxus's invention, or, if you wish, about the mathematical background of the method of finding the correct solution to (3).

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1. The main idea is to let line segments represent periods in the construction of the right-angled triangles, which implies a geometrical treatment of time. For more detail, see the figures (1,2) below.—If it is further demanded that p', q' are *not* both odd, a simple transformation is necessary: $p' = (p+q):2$, $q' = (p-q):2$. Conversely, $p = p'+q'$, $q = p'-q'$.

This heuristic process is described in algebraic terms in [1], and it shows a most interesting feature mentioned above: the auxiliary parameter *n* disappears in the synthetic part of the method. Moreover, the general view on language as a painting corresponding point to point to reality is suggested—a view that seems to be shared by Plato, too, in his semantics of time[2]. But the problem of Eudoxus's practical calculations remains to be tackled.

Details are discussed in [12]; suffice it to say here that the problem can be solved within the general framework of ancient mathematics. For we have seen that Plato's *«great harmonia»* was constructed by means of harmonious points (the ratios of division being 2 : 1 and 3 : 1), and by their means also the practical calculations can be made. The following examples are intended to illustrate these calculations.

[Moon] By means of the *arachne*, the Moon's maximum deviation from the Eudoxan ecliptic can be observed and measured in terms of tan $(\alpha/2) = q : p = 1 : 29$. Let $p : q$ now stand for the ratio of division in harmonious points, and note that calendaric considerations also suggest the «idealized» observation, for $(p+q) : p = 30 : 29$ is equal to the ratio of the «full» and «hollow» months, i.e. 30 days : 29 days[3]. Thus we have the following situation describing the combination of the second and third lunar motions (diagram not to scale), where two periods can be solved starting from one (postulated) period and from the angular measurement, the directions of rotations being determinable by the Eudoxan tradition (westward).

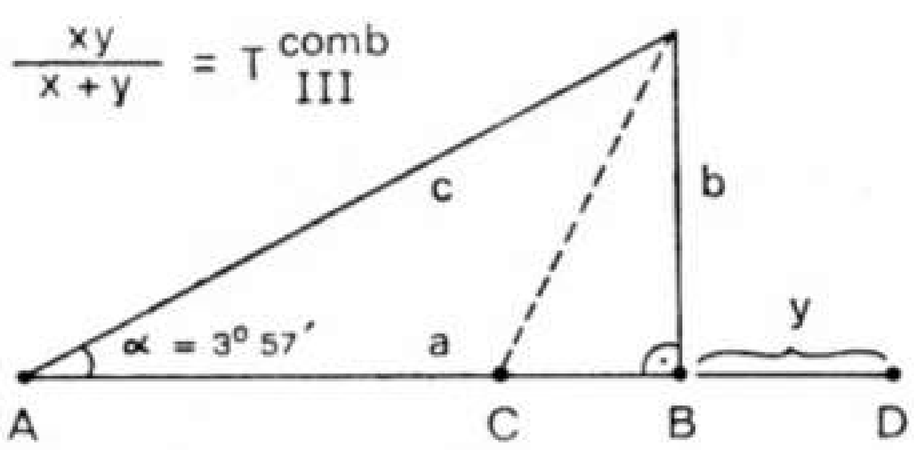

Fig. 1: The lunar triangle

The points A,B,C,D form a set of harmonious points (A,B;C,D) in the ratio of division $p : q = 29 : 1$, and $BD = y = 30$ (days) = the «full» month is postulated $= T^{comb}_{II}$, tan $(a/2) = q : p = 1 : 29$ representing an «idealized» observation by means of the *arachne*.

2. See E. Maula, *On the Semantics of Time in Plato's Timaeus*, Ann. Acad. Aboensis, Ser. B, Tom. 169.1, 1970, Abo/Finland.

3. This is obtained by simple transformations of the proportion $q : p = 1 : 29$.

Hence AB $= a = 840$ (days), AD $= x = T^{ind}_{III} = 870$ (days), CB $= 28$ (days), BE $=$ CB $+$ BD $= b = 58$ (days) $= 2T^{comb}_{III}$ and AE $= c = 842$ (days), while $\alpha = 3^0 57'$ represents the Moon's maximum deviation from the Eudoxan ecliptic, and remains within the limits of Eudoxus's observational error ($1^0 - 3^0$, according to Hipparchus's criticism). Here $a^2 + b^2 = c^2$ of course. The central role of this right-angled triangle in Eudoxus's method of analysis and synthesis is explained in [9], and its possible bearing on the emergence of the theory of stereographic projection is discussed in [11].

[S u n] The Sun, too, is credited with three motions owing to a somewhat too small numerical value given to the Eudoxan *loxotes*. It can be gathered from the Eudoxan tradition that in the case of the Sun the direction of the third (individual) solar motion is eastwards. Hence the Sun represents the second possibility as regards the sign in (2). Otherwise we follow the same procedure as in the case of the Moon. An «idealized» observation by means of the *arachne* gives the Sun's maximum (fictitious) deviation from the (Eudoxan) ecliptic, which is visualized by the same instrument, as $\tan(\beta/2) = q : p = 1 : 91$, where 91 (days) is the length of Eudoxus's equalized seasons.

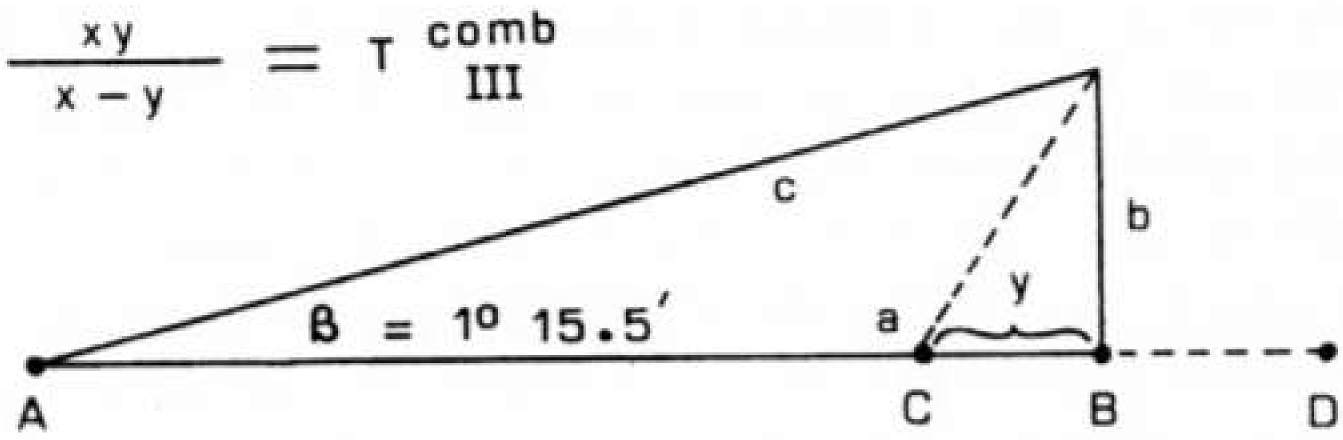

Fig. 2. The solar triangle

Hence we obtain the following situation (lines not to scale). (A,B;C,D) is a set of harmonious points, the ratio of division being 91 : 1. Here CB $= T^{comb}_{II} = 360$ (days) is postulated in advance. Hence AB $= x + y = a = 33120$ (days), $x = 32760$ (days) $= T^{ind}_{III} =$ the Sun's 'long period' implied by its «slow» Eudoxan motion, BD $= 368$ (days), BE $=$ CB $+$ BD $= 728$ (days) $= 2T^{comb}_{III}$, and $c = 33128$ (days), while $\beta = 1^0 15.5'$ represents the Sun's maximum deviation from the Eudoxan ecliptic and, being obtained by the same method as Eudoxus's other angular parameter values, speaks out strongly in favour of our reconstruction of Eudoxus's method of computation. So also does the obtained value $T^{comb}_{III} = 364$ (days), for it is well known that notwithstanding the discovery of the inequality of the astronomical seasons by Euctemon and Meton some sixty years earlier, Eudoxus equalized them. Hence $364 = 4 \times 91$ days may well be called Eudoxus's 'seasonal year', this being an example of Simplicius's use of the term y e a r in two senses, just as m o n t h stands for both 30 and 29 years[4]. As Eudoxus was interested in calendaric considerations also, there is no need to suppose that his cosmological time-reckoning contradicted the celendar. Nor does it conflict with observation either, for the 'long' solar period is too long to be observed.

The practical computations and observations by means of the *arachne* follow the same pattern in the case of the other planets also. The resulting cosmological model, however, offers some surprises, as explained in [9,11]. For the model is in full correspondence with physical reality at given times only—not always: an extraordinary temporal relation aptly characterized by the Platonic metaphor of *agalma* [4], which Plato himself uses for the model of the geocentric system in the *Timaeus*.

It is obvious, though, that the solutions (4,5) readily follow from considerations regarding harmonious points, and we are inclined to interpret Proclus's commentary on Eudoxus continuing Plato's work on the s e c t i o n (*On Eucl.* I, p. 67) as referring to harmonious points created by overlapping line segments, i.e. created by two line segments cutting each other into two in a special way. Be that as it may, we now have outlined the theoretical foundation of the *arachne*, and these are the main points of interest: the solutions to (3) in terms of $x : y = p : q$ can be given an instrumental interpretation $\tan (a/2) = q : p$; harmonius points occur both in the practical calculations and in the scales of the instrument; and an algebraic description of the heuristical aspect displays the disappearance of an auxiliary parameter in the course of the methods of analysis and synthesis.

The Ingenious Spider.

«There is a third species of this animal, pre-eminently clever and artistic», Aristotle, *Hist. Anim.* IX,39 (of the «geometrical» spider).

We have built an instrument by means of which one can measure precisely those observational parameter values that Eudoxus needed in his astronomy (Fig. 3). Its construction does not in any way exceed the ancient technician's skills[5].

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4. Moreover, Eudoxus's 'seasonal year' is another example of his deliberate «theoretical errors», made for methodological reasons.

5. A model for demonstrative purposes (scale 2 : 1) was built at the workshop of Physics, University of Oulu, by Karl Sandman, Kauko Hägg, Penth Tiitta and some students of the history of science in the spring of 1976.

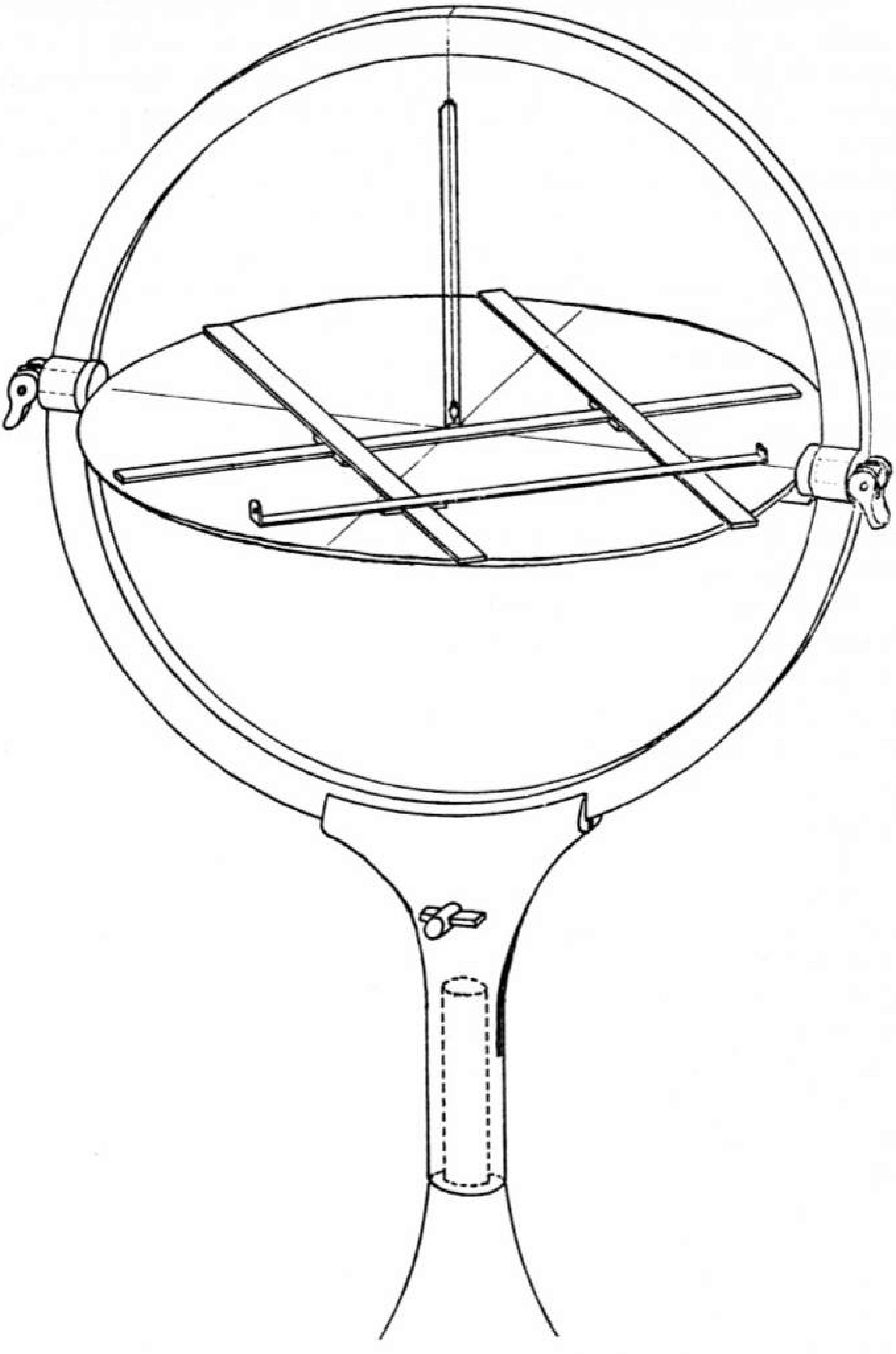

Fig. 3. The *arachne*, general view.

The instrument consists of four main parts. (i) A stand permitting rotation about two ortogonal axes supports (ii) a meridian circle adjustable and lockable at any position in two dimensions and supporting in turn (iii) a circular plate rotating about the diameter of the meridian circle, with respect to which the plate can be adjusted and locked. Alternatively, the circular plate can be attached to the stand, in which case the circle may rotate about the diameter of the plate. On top of the plate (iv) the measuring unit resembling the sliding callipers turns about a *gnomon*. In addition there is a plumb hanging from a string (stathmēs) for purposes of calibration and to demonstrate certain angles. If the measuring unit is removed, the instrument looks like a big bronze mirror, and one may well call it an *enoptron* after a lost work of Eudoxus. In the measuring unit and the plumb-string one may perhaps see a spider and its thread, and hence call the whole instrument an *arachne*. We presume that if there really was an instrument or a cosmological model on Plato's table (see Cornford's *Plato's Cosmology*, London-New York 1971 [1937], p. 74 ff.) it might have been of this type.

On the circular plate two scaled axes are grooved (in addition to certain curves, perhaps, which we shall discuss below); there are thirteen units in all four directions measured from the centre to the perimeter. In the main trunk of the measuring unit and in its two arms there are both unit scales and also others based on harmonious points [2,3]. An auxiliary trunk, moving parallel to the main trunk, has no scales but is provided with sights *(dioptra)*.

The measuring unit perhaps merits a separate picture (Fig. 4). It is not necessary to engage in entomological debates on the precise number of a spider's feet. Suffice it to say here that while the main trunk (0) and the two arms (M,N) at right angles to it are essential in all usages, the auxiliary trunk (P) is only needed in the demonstration of a parallel to the main trunk. As this could be done by means of the string or a separate ruler, too, (P) is perhaps superfluous. In any event the motions of the parts are indicated by small arrows. (M) and (N) move parallel to one another, and can be locked with respect to (0). Finally, it is advişable to attach a runner (R) to both (M) and (0) so that the string may run via them.

In fact, the measuring unit may be conceived of as consisting of four *gnomons*, and its correct place in the history of astronomical instruments is between an ordinary *gnomon* and a cross-staff or Jacob's staff. By means of the *arachne*, however, observations and calculations can be executed which exceed the capacities of these kindred instruments. It may be noted also that although the *arachne* combines certain functions of rulers and compas-

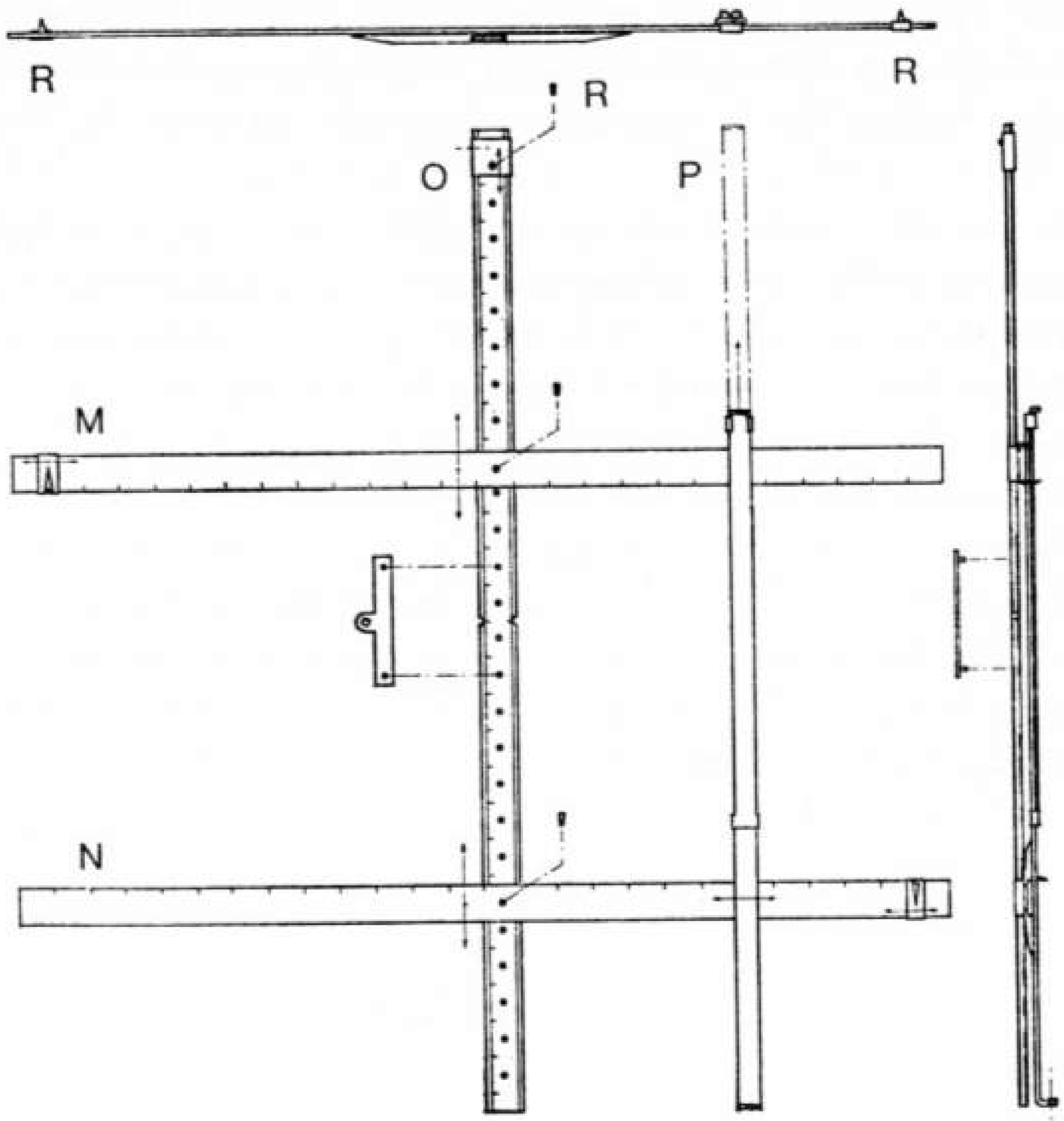

Fig. 4. The *arachne*, the measuring unit

ses, it preserves also the character of a cosmological model and simultaneously a real instrument permitting heuristic experiments in a sense that escapes the rules and compasses in the axiomatized Greek geometry. Its more ordinary usages, and the obvious usages as a *gnomon* or a sun-dial, we shall not discuss.

The Angles of the Heaven.

It is possible to measure the angle between any two visible celestial or mundane objects or the angle between an object and any of the great celestial circles, by means of the *arachne,* but we demonstrate an even more representative measurement in ancient astronomy, the measurement of a star's angular distance from the horizon. Eudoxus is reported to have observed Canopus (α Carinae) in Cnidus (Strabo C 119 after Poseidonius). Hipparchus (*In Arat.* p. 114,20-8 Manitius) says that Eudoxus put Canopus exactly on the «always invisible circle» and that this was not correct because Canopus was invisible in Cnidus (lat. 36°43′). Hipparchus's criticism shows that he had not yet discovered precession, for in fact Canopus was barely visible in Cnidus in Eudoxus's time (its declination was —52.8⁰ according to U. Baehr's tables *Tafeln zur Behandlung chronol. Probleme*, Karlsruhe 1955).

In the period between Eudoxus and Hipparchus, the positions of the stars relative to the celestial sphere had been changed due to the precession, [8, p.10] Eudoxus's special concern with Canopus is probably connected with his attempt at an estimate of the diameter of the Earth, for Canopus is an easily recognizable object which Eudoxus must have observed even in Egypt.

Presupposing that the two observation places are on the same longitude, one can compute from two such observations their angular distance[6] measured in parts of the whole meridian (and if their distance is measured in stadions, the meridian's length in stadions can be computed, too). And it is possible that just the meridian circle of the *arachne* has given rise to the idea of the division of the circle.

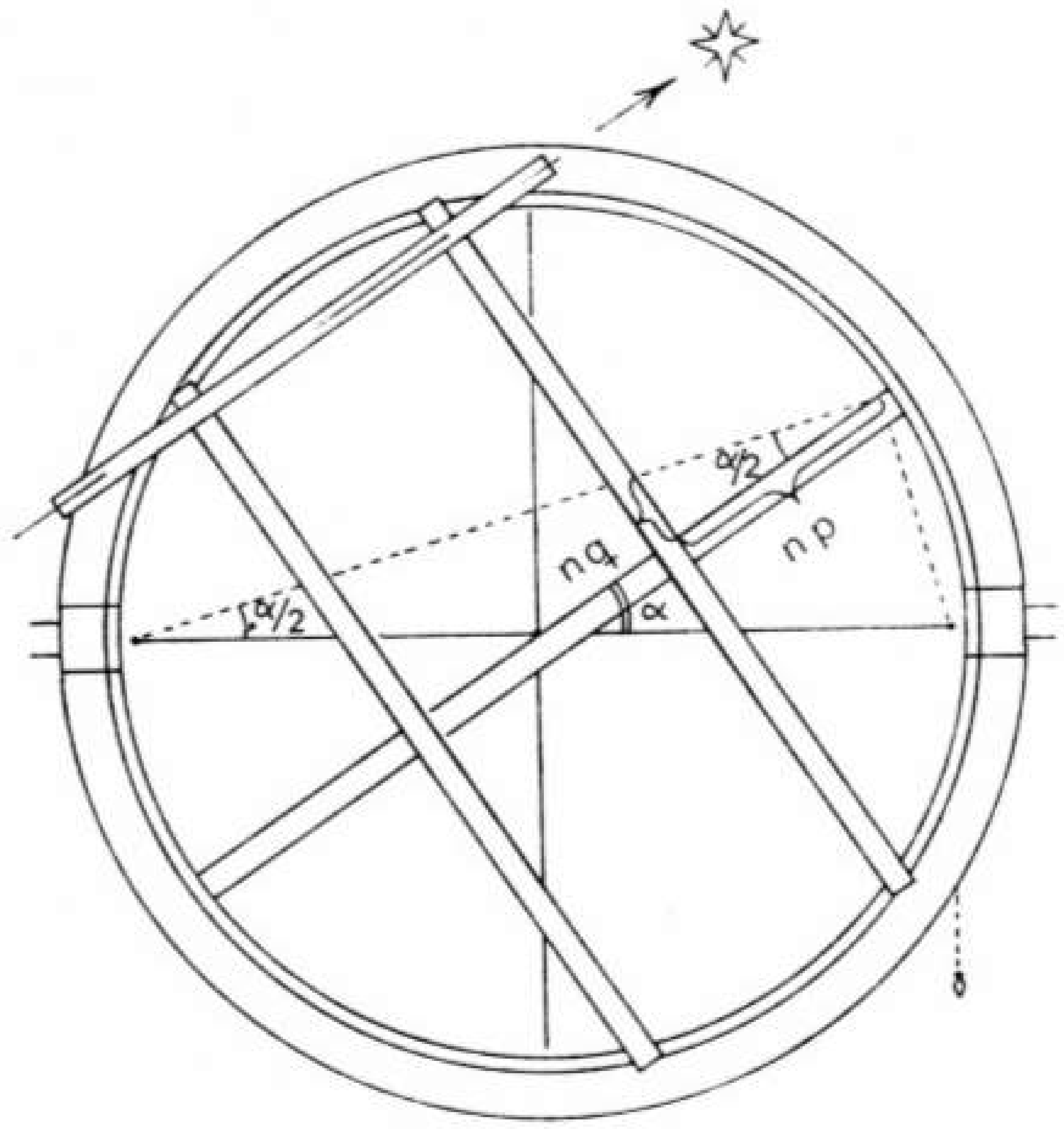

Fig. 5. The *arachne*, measurement of a star's angular distance from the horizon.

Be that as it may, the observation technique is shown in the attached figure (Fig. 5). The observation is made by means of the dioptra, and

6. Because Canopus is put on the antarctic circle, the angular distance between the Cnidian and Egyptian observation sites would be equal to Canopus angular height

the angular distance from the horizon appears as α, the acute angle at centre. The corresponding angle at the circumference ($\alpha/2$) can be demostrated by the string, and tan $(\alpha/2) = q : p$ read from the scales of the measuring unit. Since the arm at right angles to the main trunk can be moved with respect to the trunk, it is quite likely that in some position the ratio $q : p$ corresponds to the prefixed lines of the two scales. If indeed the division of the circle was developed in this way, then, since tan $(a/2) = q : p$ is a ratio of relatively prime numbers, the circle may first have been divided into eight (because tan $90^0/_2 = 1 : 1$) or into twelve equal parts (because sin $60^0/_2$ $1 : 2$). This idea is corroborated also by Eudoxus's method of exhaustion.

Eudoxan Shadows and Hours.

It was customary in the Hellenistic period to indicate the geographical latitude of an observation site by speaking about either (i) the ratio of the length of the longest day of the year to the shortest night or (ii) the ratio of the two parts of the tropic divided by the horizon at the summer solstice. When the well-known Hellenistic method for the determination of the geographical latitude is used, these two locutions become synonymous. The presuppositions of this development are discussed in [7], where it is also shown that these two locutions could not be considered synonymous before the equalization of hours in the Hellenistic period, and certain other conventions. For Eudoxus they must have meant two different things.

Now it is known that Eudoxus gave two ratios of the tropic divided by the horizon at the summer solstice, $K_{ss} = 5 : 3$ and $K_{ss} = 12 : 7$ (Hipp. *in Arat.* I, 2.22, 3.9.). It is shown in [7] that if exactly the same value for the obliquity of the ecliptic is used as the one which we deduced from Plato's *Timaeus,* these two ratios correspond to the latitudes of Babylon and Egypt (the ruins of Babylon and Alexandria). The accuracy is even better than that obtained by the later Hellenistic method (the error being some 16′). The principles of the reconstructed Eudoxan method are illustrated (in the case $Kss = 5 : 3$) in the attached figure (Fig. 6). In modern terms, if $R = 12 =$ radius of the summer tropic, in the case $Kss = 5 : 3$, then $P'Q = R\cos\beta$, $OP' = R\tan a$, and $\tan\varphi = QP' : OP' = R\cos\beta : R\tan a$. Hence $\cos\beta = \tan\varphi \tan a$.

It is seen that the use of this method presupposes that the celestial pole

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from the horizon in Egypt. But a further step is needed to interpret this in terms of parts of the circle. Besides the measured distance between Egypt and Cnidus must be replaced by the same multiplied by the cosinus of their angular longitudinal distance.

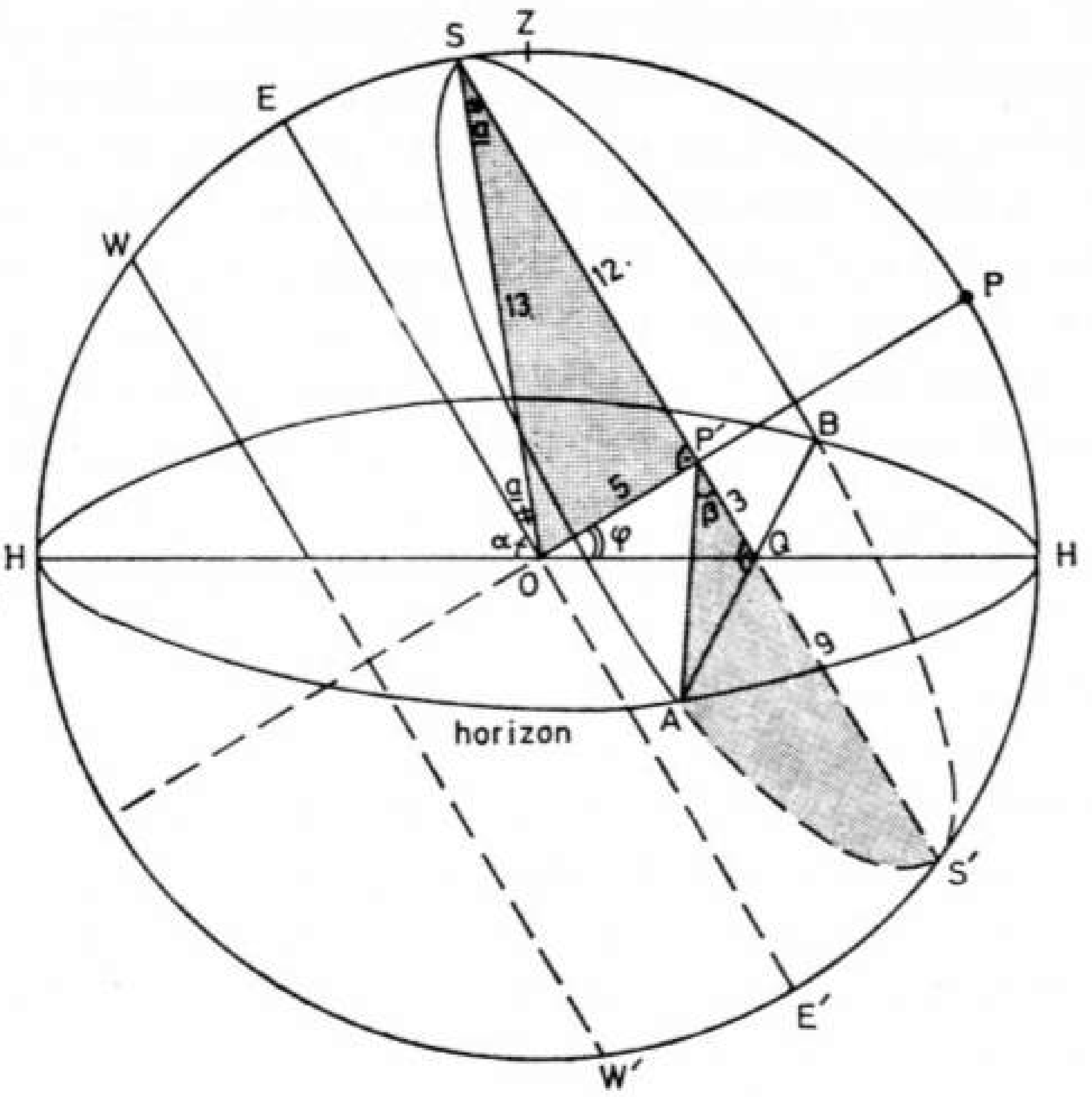

Fig. 6. The celestial sphere with observer at O, horizon (HH), diameters of the winter tropic (WW'), equator (EE') and summer tropic (SS'), declination of the equator to the horizon at the equinox (α), the auxiliary rectangular triangle which Eudoxus probably used (SOP') and which determines the value given to the obliquity of the ecliptic (a), half the night arc (β) corresponding to the division of the summer tropic on the point Q in the ratio 5:3 (i.e. $K_{ss} = 5:3$) at the summer solstice. Parts of the diagram below the horizon marked with dotted lines.

P can be determined at the summer solstice. How this is accomplished by means of the *arachne* is shown in the next figure (Fig. 7). The ratio K_{ss} $= a : b$ can be read from an ordinary scale on the arm because the instrument indicates both the horizontal plane, diameters of the tropics, the obliquity of the ecliptic, and the celestial pole at the same time.

The value of this argument gains in strength because Hipparchus's remarks in his commentary on Aratus are independent of the other traditions partaining to Eudoxus's theory of the homocentric spheres. Moreover, Hipparchus seems to be drawing on Edoxus directly, repeating his results as they stand and not trying to reconcile them with later observations, for he obviously does not know Eudoxus's method[7]. A similar case is Hippar-

7. Owing to a different method, Hipparchus's interpretation of K_{SS} differs from that of Eudoxus's, although even in Hipparchus method we obtain a formula of the type $\cos\beta = \tan\varphi \tan\varepsilon$; see [7]. But in Hipparchus β is half of the night-arc, half of the

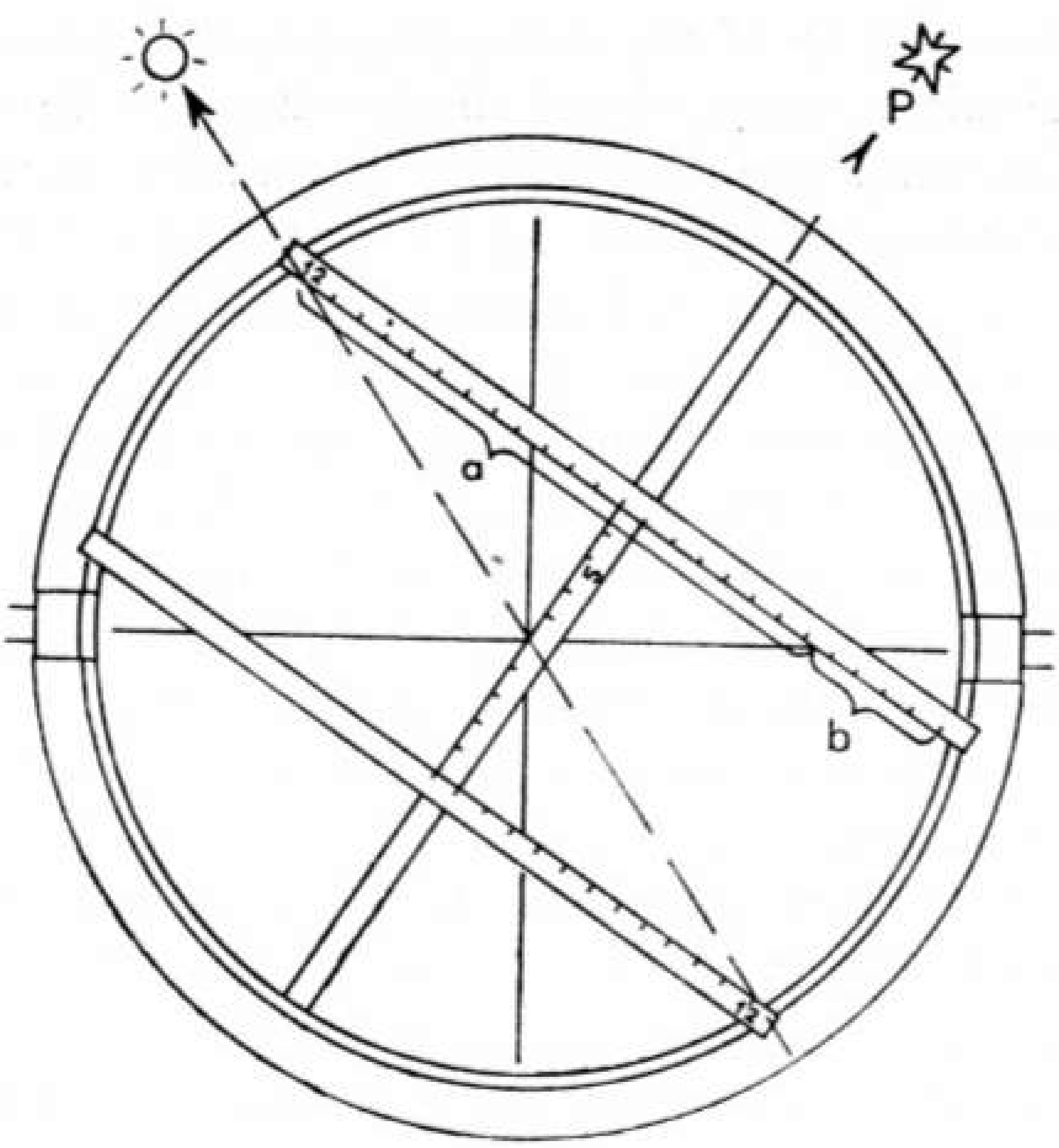

Fig. 7. The *arachne*, the division of the tropic by the horizon at the summer solstice (a:b)

chus's wrong conclusions from Pytheas's *gnomon* value $G_{SS} = 120 : 41\ 4/5$ for Massalia. Hipparchus's surprising miscalculations (error 2^0) suggest that he had access to data computed in a way quite different from his.

It is most interesting to observe in this and in the previous case the change of the point of view. This is further manifest in the two ways of attaching the *arachne* to its stand either for meridian observations or else for observations pertaining to the other great celestial circles. But because the *arachne* is not only a model of the geometric system, but also a mathematical instrument, the choice of the point of view enables one to measure a number of planetary motions with respect to the great celestial circles also. Actually this freedom of choice of the point of view is anticipated in Plato's Myth of Er in the *Republic* X, where the geocentric system is viewed from above.

The Geometrical Score of the Harmony of the Spheres.

Owing to the scales based on harmonious points (the rations of divi-

angle at which the night-side of the earth is seen from a point directly above the north pole, while in Eudoxus it is half of the arc of the tropic below the horizon at the summer solstice seen from the centre of the plane of the tropic (point P′ in Fig. 6).

sion being 2 : 1 and 3 : 1), of the measuring unit, the a r a c h e can also be used in determining proportional angular distances between stars. In this type of observation one tries to find quadruples of starts on a line corresponding to sets of harmonious points on the scales of the instruments.

At least two motives can be suggested for this type of observation: (i) an astronomer, especially one associated with the Academy, may wish to show that the starry heavens are constructed according to principles of geometrical harmony (which does not exclude the musical harmony, as can be seen from the musical intervals used in Plato's «great harmony»), and (ii) an astronomer operating without proper star-maps, and hence obliged to refer to fairly inaccurate and changing descriptions of the constellations, may wish to specify and standardize these descriptions starting from principles of geometrical harmony.

Looking at the extant material pertaining to Eudoxus's (lost) books *Phaenomena* and *Enoptron* (fragments 1-120 in Lasserre's *Die Fragmente des Eudoxos von Knidos,* 1966), one can hardly doubt that Eudoxus was occupied with problems of specification and standardization although, as is the case elsewhere too, only some of his results are known while his method must be reconstructed from these. In uncertain cases, though, for instance when the location of a special star in one or another constellation (with respect to the great celestial circles or the tropics) must be decided upon, it is quite natural to apply geometrical principles. Suppose that three stars unquestionably belong to a constellation and form three points of a set of harmonious points. A fourth star corresponding to the fourth harmonious point on the scale, may then be grouped together with the previous ones. If one consults modern star-maps and the list of stars mentioned by Eudoxus, this procedure suggests itself. It is worthwhile, therefore, to show that harmonious points can indeed be defined in terms of Eudoxus's contribution to Euclid's *Elementa,* Book V.

Let us consider similar triangles of two different kinds (Fig. 8). Suppose that the line segments EC, CF are equal. Then on the one hand the triangles ACE and ADG, and on the other hand the triangles CFB and BGD, are similar. Hence the points A, B, C, D form a set of harmonious points (A, B; C, D). In order to derive the condition for harmonious points, we now take any equimultiples of the line segments AD and EC, and any equimultiples of the line segments AC and GD. Let the former equimultiples be, for instance, halves of, and the latter

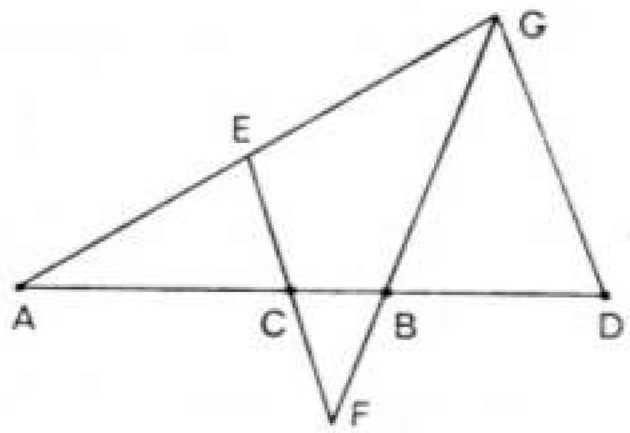

Fig. 8. Harmonious points

equimultiples double, the corresponding line segments. Then the equilmultiple of AD falls short of that of EC, and the equimultiple of AC falls short of that of GD. Now, by *Elem.* v, Def. 5, we obtain the proportion (1) AC : AD = EG : GD and the proportion (2) CF : GD = CB : BD. Since EC = CF, we obtain further by *Elem.* v. 7 the proportion (3) EC : GD = CF : GD. From (1) and (3) we obtain by *Elem.* v. 11 the proportion (4) AC : AD : CF = GD, and from (2) and (4) by the same proposition the proportion (5) AC : AD = CB : BD. Finally, from (5) we obtain by *Elem.* v. 16 the proportion (6) AC : CB = AD : BD. This is the condition under which the points A, B, C, D form a set of harmonious points (A, B; C, D).

As for the interrelations of the line segments partly overlapping on the scales of the measuring unit and the angles, these are best represented by means of the line segments and the tangents of the angles (for tan $\alpha =$ the ratio of the *gnomon* to its shadow). We obtain several interrelations which are illustrated by the attached figure (Fig. 9). We list some of them below.

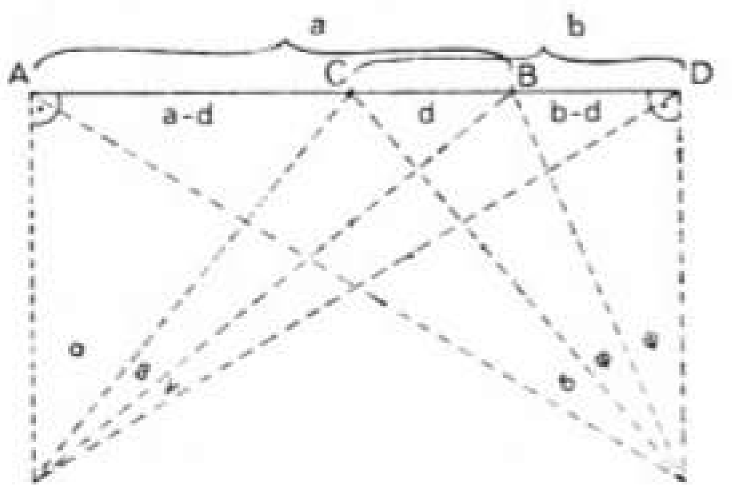

Fig. 9. The interrelation of angles and harmonious points

(i) tan $(a+\beta)$: tanα = a : (a-d), where the coefficient $(a-d)^2$: ad gives the ratio of division, (ii) tan $(\alpha+\beta+\gamma)$: tan $(\alpha+\beta)$ = (a+b—d):a, where the coefficient is a:(b—d), (iii) tan $(\alpha+\beta+\gamma)$: tan α= (a+b—d):(a—d), where the coefficient is (a—d):(b—d), (iv) tan $(\varphi+\psi)$: tan φ=b:(b—d), where the coefficient is (a+b—d):b, (v) tan $(\delta+\varphi+\psi)$: tan $(\varphi+\psi)$ = (a+b—d):b, where the coefficient is, contrary to the previous case, b:(b—d), and finaly (vi) tan $(\delta+\varphi+\psi)$: tan ψ= (a+b—d):(b—d) or the ratio of division of the line segments, the end-points of which create the set of harmonious points (A, B; C, D).

It is interesting to note that if the ratio of division is denoted as p:q, then b:a = $2pq:(p^2-q^2)$ = tanξ, while q:p = tan $(\xi/2)$, where ξ is an acute angle and a,b the two shorter sides, of a right-angled triangle. This creates a connection between harmonious points and the solutions (4,5).

For whereas in terms of the line segments partly overlapping the overlapping part $d = \frac{a+b-\sqrt{a^2+b^2}}{2}$, in terms of the right-angled triangle again $\sqrt{a^2+b^2} = c$ or the hypotenuse. Hence $d = \frac{a+b-c}{2}$, and the ratios of the tangents (i-vi) can be interpreted in terms of the sides of the right-angled triangle. But, furthermore, by means of the solutions (4,5), they can be interpreted also in terms of periods. Perhaps we have here an example of Eudoxus, who «was the first to increase the number of the so-called *general theorems*» (Proclus, *On Eucl.* I, p. 67).

Irrationality and Invention.

It is well known that the ancients had discovered not only theoretical geometrical solutions but also practical instrumental solutions to the problem of the two mean proportionals, which is a : x = x : y = y : b. The main sources are Pappus's *Collectio,* Book 3 (ed. F. Hultsch, Berlin 1876-8, Band 1, pp. 56-64), and Eutocius's commentary on Archimedes's *De sphaera et cylindro,* Book ii, Prop. 1, appearing in J. L. Heiberg's critical edition of Archimedes's *Opera* (Band 3, Leipzig 1915^2, pp. 54-106). The whole tradition seems to derive from Eratosthenes's Platonicus, and through him from Eudemus (see Lasserre, op. cit., p. 163 ff.).

We are especially interested in the so-called Platonic solution and Eudoxus's solution. The former (instrumental) solution is discussed by J.E. Hoffmann in *Über die sog. platonische Konstruktion von Kubikwurzeln,* «Sudhoffs Archiv» 58 (1974), pp. 60-63. But by means of our reconstruction of the *arachne,* too, square and cubic roots can be extracted. We submit, therefore, that the so-called Platonic solution to a : x = x : y = y : b and Eudoxus's solution are identical, albeit the *arachne,* in addition to illustrating the proof, also suggests the method of invention preceding the proof.

It is true that Heath (*A History of Greek Mathematics,* 1, pp. 255-258) concluded that the so-called Platonic solution was invented in the Academy by someone contemporaty with or later than Menaechmus, Eudoxus's pupil. But Heath's argument depends on the fact that in analytical terms the former solution, being found by means of a curve of the third degree, is more difficult than that of Menaechmus, which is found by means of the intersection of curves of the second degree (either two parabolas or a parabola and a hyperbola). It is not the Cartesian analytical treatment, however, but the use of the instruments[8], which must be considered here. For Plutarch (*Marcell.* 14, p. 59 f., Sint.) tells us that Eudoxus used «machines» constructed on the basis of geometrical theories, especially on the theory of the two mean proportionals (see RE, s.v. *Eudoxos,* § 4). Besides, Plato (at *Tim.* 32b)

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8. Or, in other words, it is the simplicity of the condition of the modern equations, in terms of the use of the *arachne,* which determines the «degree of difficulty» of the solutions.

seems to refer to the problem of the two mean proportionals and to its solution as to something well known.

In addition to the use of an instrument, two main points emerge from the tradition pertaining to Eudoxus's solution (D 24-29 in Lasserre). First, Eudoxus used «curved lines» (καμπύλαι γραμμαί) in the d i s c o v e r y of the solution, but did not refer to them in the p r o o f. And second, Eutocius criticizes Eudoxus for having confused a discrete proportion (e. g. $a : b = c : d$) with a continuous one (e.g. $a : b = b : c$). Earlier commentators have never succeeded in combinig these two features (see Lassere, op. cit., pp. 163-6, and Heath, op. cit., i, pp. 249-251). Yet both can be understood when the use of the *arachne* is considered.

Leaving aside its more obvious use in the extraction of square roots, or the finding of one mean proportional, we will concentrate upon the cubic roots. The fact that even square roots can be extracted, this being a special case of the more general problem of finding two mean proportionals, should be remembered, however. For it is from the connection between the problems of one mean proportional and a δύναμις taken in the sense of a quadratic value of a rectangle (see Arpád Szabó, *Anfänge der Griechischen Mathematik*, München-Budapest, 1969) that we have continued our studies pertaining to the dynamic world-view in [12]. A mediating step is the reconstruction of Eucoxus's *arachne*, as we shall see presently.

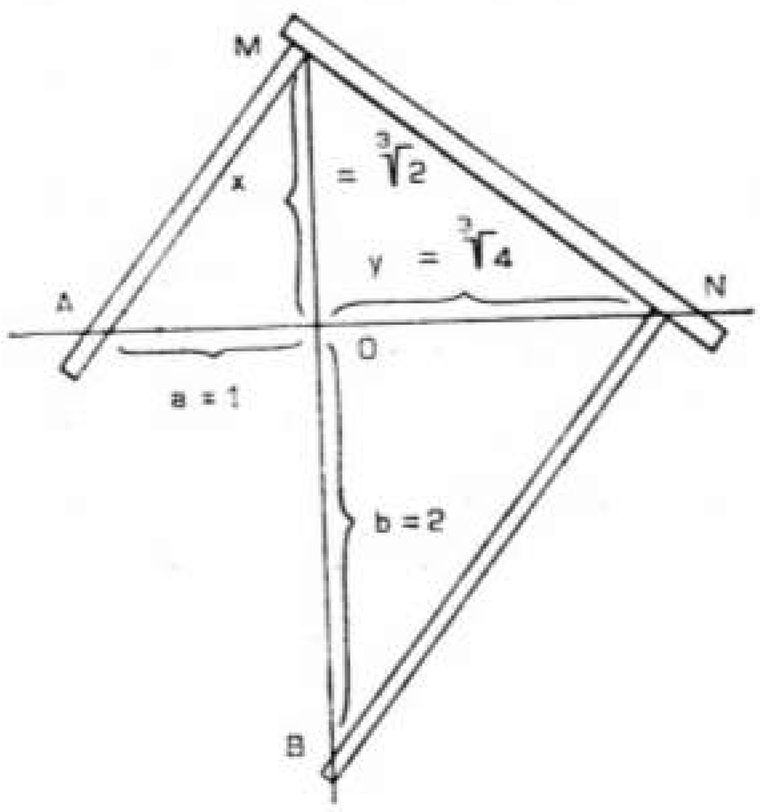

Fig. 10. The Platonic solution

Let us outline first the p r o o f of the so-called Platonic instrumental solution, which is possible as soon as the instrument has reached a desired position. If $a=1$ and $b=2$, say, the solution to $a : x = x : y = y : b$ appears as in the attached figure (Fig. 10).

Because the angles AOM, MON, NOB, AMN and BNM are right angles, the triangles AMO, MON and NOB are similar, and their respective sides proportional. Hence it can be proved (without any «curved lines») that, in modern terms, $1 : \sqrt[3]{2} = \sqrt[3]{2} : \sqrt[3]{4} = \sqrt[3]{4} : 2$. In other words, one has obtained by means of the *arachne* two approximations: $x \approx \sqrt[3]{2}$, $y \approx \sqrt[3]{4}$.

But how will the desired position of the instrument be found? In other words, what is the heuristic insight succouring the proof? Speaking abou̧ the connection between Menaechmus's solution and the so-called Platonic solution by means of a «machine» consisting of two right-angled rulers, Heath (op. cit., i, p. 257) notes: «That it is possible for the machine to take up the desired position is clear from the figure of Menaechmus, . . ., although to get it into the required position is perhaps not quite easy». We can, however, suggest even two procedures leading to the required position, and the latter one of them explains the alleged confusion of a disccrete proportion with a continuous one, while both explain the use of «curved lines» occurring in the heuristic part but disappearing in the proof. It may be added that these methods were found in a «practical» way, i.e. by the use of the reconstructed instrument. The analogy between these heuristic ideas and at striking disappearance of the auxiliary parameter n in dealing with the combinations of spherical motions in the theory of homocentric spheres by means of the methods of analysis and synthesis [1,9] is conspicuous, however. It would seem that Eutocius, by chance, refers to the very hallmark of Eudoxus's methods of invention.

The Spider's Two Dances.

In the first method the measuring unit is released from the *gnomon*, and the plumb-string (but not the auxiliary trunk) of the *arachne* is needed. We illustrate the steps of the method by simplified diagrams.

1⁰. One makes the initiative (first) guess (x_1), i.e. one makes the arms of the instrument go through the points A, B and keeps the apex M_1 of the right angle AMN on the axis of the *enoptron*. Now the apex of the other right angle, N_1, does not meet the other axis (Fig. 11). A perpendicular is dropped (by means of the plumb-string) from N_1 to M_1B. From similar triangles $AO:M_1O = M_1O_1:N_1O_1 = N_1O_1:BO_1$ or, in other words, $AO:x_1 = (x_1 - e_1):y_1 = y_1:(BO+e_1)$, where $e_1 = OO_1$ is the first error term. Thus x_1 is too great by e_1, as compared with the final, required solution.

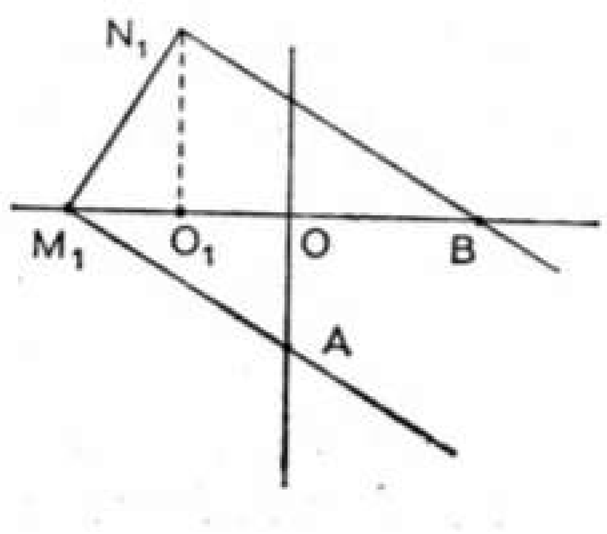

Fig. 11

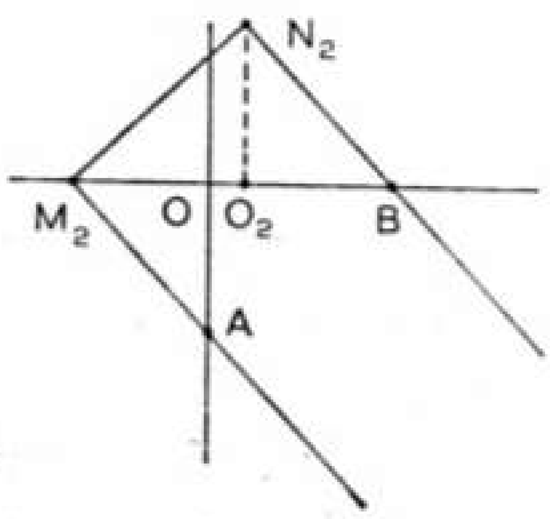

Fig. 12

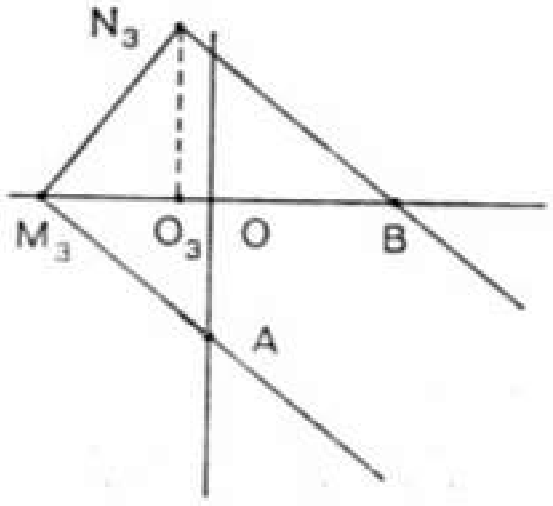

Fig. 13

2º The second guess is $x_2 = x_1 — e_1$. The situation is shown in the figure (Fig. 12). From similar triangles $AO:M_2O = M_2O_2:N_2O_2 = N_2O_2:BO_2$ or, in other words, $AO:x_2 = (x_2+e_2):y_2:=y_2:(BO—e_2)$, where $e_2<e_1$ is the second error term. Thus x_2 is too small by e_2.

3º The third guess is $x_3 = x_2+e_2$. The situation is shown in the figure (Fig. 13). From similar triangles $AO:M_3O = M_3O_3:N_3O_3 = N_3O_3:BO_3$ or, in other words, $AO:x_3 = (x_3—e_3):y_3 = y_3:(BO+e_3)$, where $e_3<e_2<e_1$ is the third error term.

4º The next guess is $x_4 = x_3—e_3$. The same procedure will, in principle, be repeated *ad infinitum* until $e_\infty = 0$ andt he so-called Platonic solution is achieved.

These are the main points to be observed. First, the same rationale is discernible at every step of the procedure (the use of similar triangles), and it is the same in the final proof-situation also. Second, the method converges whatever the initial guess. Third, the procedure can indeed be indicated by means of a «curved line», on which the points N_n are situated. Noting that if, for instance, AO = 1 and OB = 2, then $1 : OM = (x+OM) : y = y : (2 — x)$, whence $OM = (y^2 — 2x + x^2) : (2 — x)$, the equation of this «curved line» (of the fourth degree) can readily be obtained. Fourth, at each step of the procedure, when the proportionality of the sides of similar triangles is leaned upon, a discrete and a continuous *analogia* are equa-

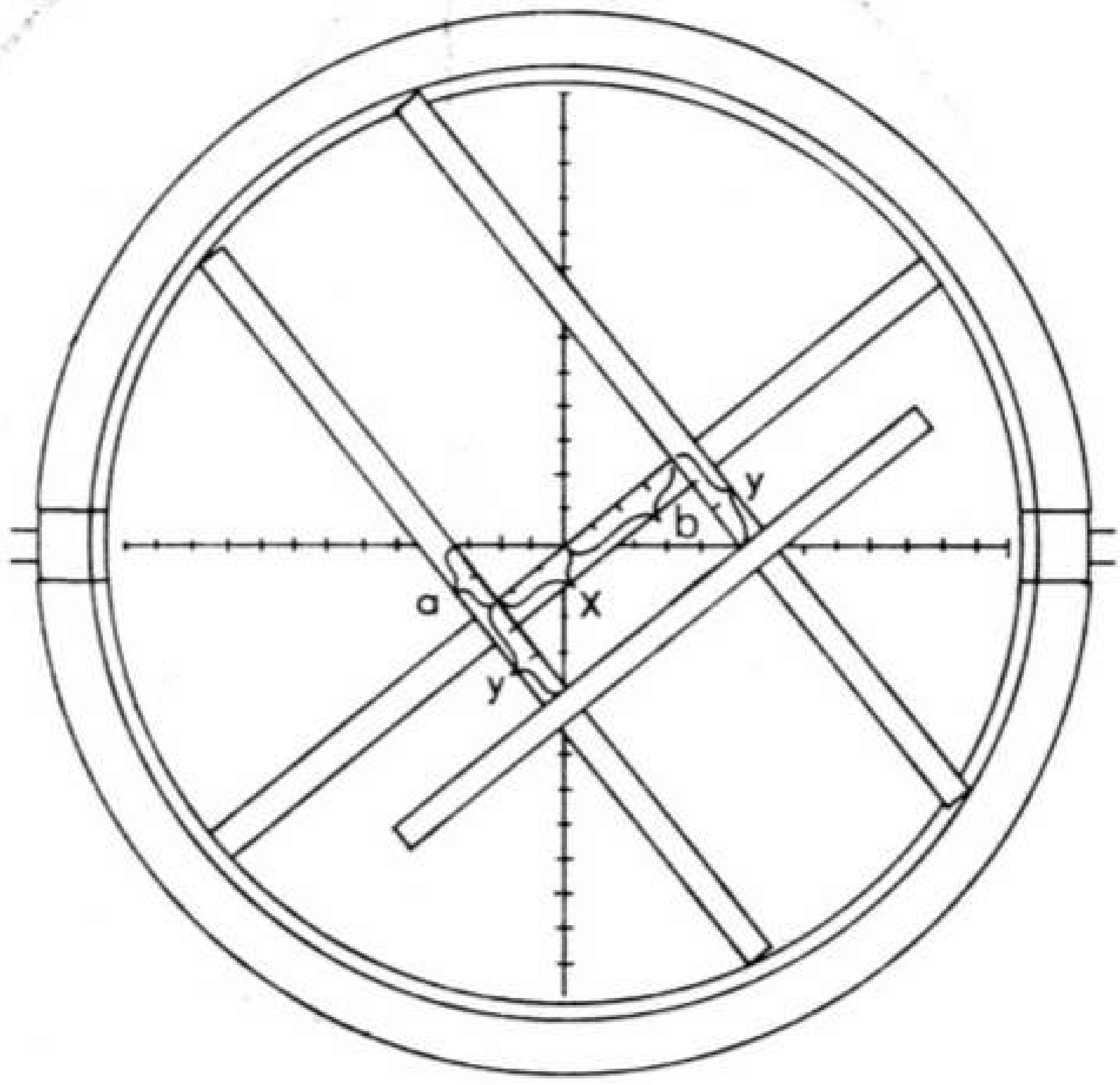

Fig. 14. The *arachne* used as a computer.

lized. Fifth, the whole method of invention with its increasingly more accurate approximations from above and from below, is remarkably akin to the known Pythagorean method of approximation to surds by algorithmic means. The discovery of this relationship seems to prove beyond reasonable doubt that one of Eudoxus's methods indeed was of the type adumbrated here. Its implications for mathematical heuristics and theory-formation are far-reaching. They pave the way for the epistemology within a dynamic world-view, which is discussed in [12].

In the second method the *arachne* turns about the *gnomon* on the *enoptron*, and the auxiliary trunk is useful (although perhaps not quite necessary). Suppose that we are trying to find the two mean proportionals between two line segments a, b. Now a is kept in all positions as the distance from the horizontal axis of the *enoptron*, measured at right angles from the main trunk of the measuring unit, and b is fixed on the scale of the main trunk, measured from the *gnomon* onwards. (Fig. 14). Thus the end-point of b describes a circle, and the end-point of a describes a «curved line» (Fig. 15), while the other end-point of a follows the axis of the *enoptron*. In this case the desired position is obtained more «mechanically», i.e. as soon as a rec-

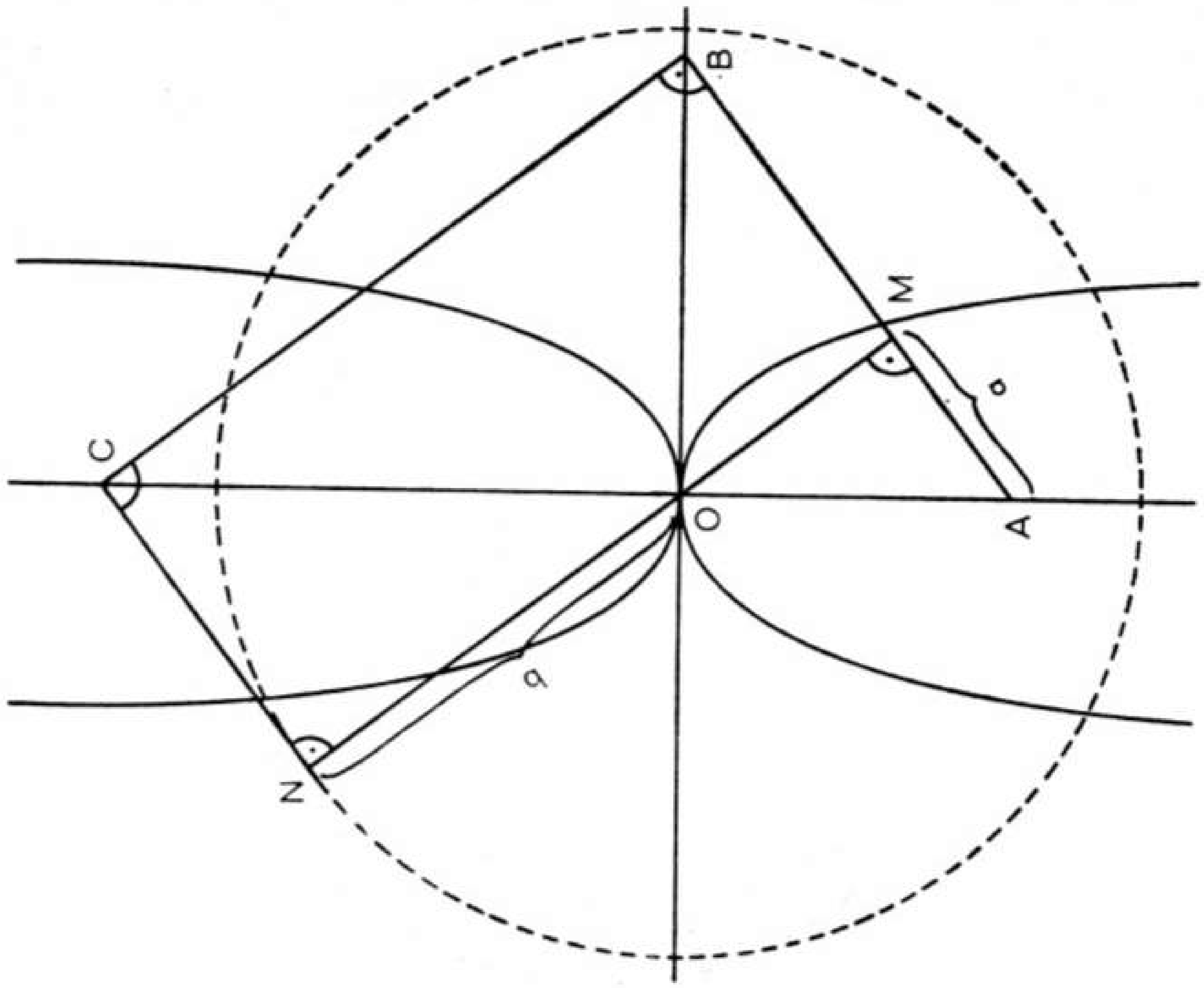

Fig. 15. The second «curved line».

tangle (completed by means of the auxiliary trunk, say) is obtained. The rectangle is MNCB.

If one makes a=1, b=2 the conditon for the «curved line» is $x : y = (\sqrt{x^2 + y^2}) : 1$, whence the equation (of the fourth degree) can readily be obtained.

It is most interesting to note that when the distance a=1, and the ensuing curve is drawn or engraved on the *enoptron,* not only the cubic root of two, but all cubic roots can be extracted by means of it. Hence this curve ($x^2{:}y^2 = x^2+y^2$) enjoys of special status, and even its basic condition («distance = 1») is as simple as ever can be expected. Moreover, if the plumb-line is used, even the auxiliary trunk may be discharged, and the traditional position of the so-called Platonic solution is obtained (Fig. 16).

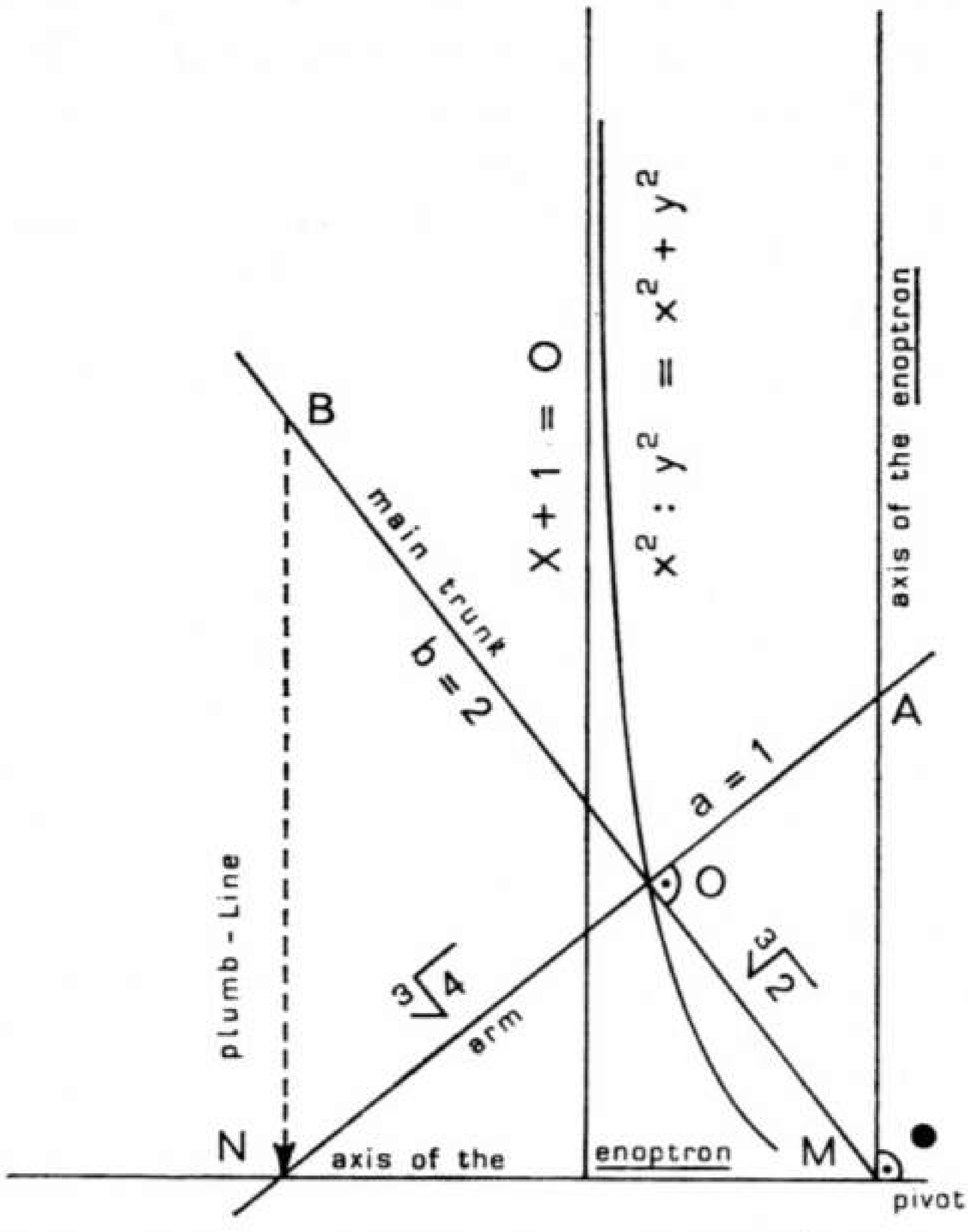

Fig. 16. The Platonic solution by means of the *arachne* and plumb-line (cf. Fig. 10).

The second method adumbrated exhibits some similarity to Eudoxus's way of representing the apparent planetary motions by means of combined spherical rotations. It is not obvious, however, how Eutocius's remark on discrete and continuous proportions could be accounted for. But Plutarch's fits nicely here. And we cannot be very far from the birth-place of Menaechmus's discovery of the curves we call the conic sections, either. Note, for instance, that the second «curved line» approaches asymptotically the lines $x \pm 1 = 0$. But this observation already points toward far more general ideas.

Reflections upon the Mirror.

In order to see how the *arachne* reconstructed ushers in a dynamic world-view, it is wise to recapitulate the argument amounting to the reconstruction. The recapitulation may stand for a conclusion, and the subsequent notes for a plan of future research.

We started from Plato's *«great harmonia»* in the *Timaeus* on the working hypothesis that the edifice of the world-soul is erected either with Eudoxus's cosmological system as its frame of reference or else giving an impetus to Eudoxus's system. We then discovered the obliquity of the ecliptic (from $\sin \varepsilon = 5/13$) used by Plato and, presumably also by Eudoxus, for the value obtained is somewhat smaller than the real value, but so it should be, indeed, in Eudoxus. Plato's harmonic intervals show also that *Pythagorean triples* are generated (from $p=3$, $q=2$), and, that both acute angles and planetary periods are expressed in terms of them [8]. Moreover, the *«great harmonia»* implies that Plato made use of harmonious points created by overlapping line segments. For if two line segments are equally divisible by the overlapping part, the ratio of division is either $2:1$ or $3:1$, i.e. the basic double and triple intervals in Plato, and conversely [2,3].

That the *loxotes* obtained was indeed used by Eudoxus, was confirmed by a study in Eudoxus's method of determining the geographical latitude, for using the value obtained two surprisingly accurate latitudes (Babylon and Egypt) could be computed [7].

The use of this additional information made it possible to reconstruct Eudoxus's method of computation in his planetary theories and, moreover, his methods of analysis and synthesis and an essential feature of his heuristical method [1,9]. Eudoxus's central problem being the combination of two spherical motions at a time, the ensuing generalized proportion to be solved is, in terms of two unknown and one known rotation period, $xy:(x \pm y) = T^{comb}\, n:n$, where n is an auxiliary parameter that disappears in the synthesis [9,10]. The *solution* to this generalized proportion combines rotation periods with acute angles and relatively prime integers generating Pythagorean triples where the corresponding triangles have the same rôle as auxiliary drawings in geometrical proof. For $y : x = q : p = \tan(\alpha/2)$, where the whole angle α represents a planet's maximum deviation from a given plane, e.g. from the Eudoxan ecliptic.

At this stage we observed a further link to Plato's world-soul. For both the ensuing Eudoxan system and the world-soul, which Plato compares to an *agalma* [4], display the same characteristics of models that fully correspond to physical reality at given times only. This discreteness notwithstanding the reconstructed Eudoxan system fully explains, starting from the known Eudoxan periods, more than one hundred other main parameters of his cosmology. The practical calculations can be made with the aid of harmonious points, combined with actual (or idealized) observation.

Knowing, then, that in the solution to the problem of combined spherical motions Eudoxus obtained acute angles measured by the tangent of the half angles, we have rebuilt the *arachne* which so measures the angles. Moreover, it can be used in determining the geographical latitude in the way Eudoxus proceeded, and in the extraction of square and cubic roots. In the closer analysis of the last-mentioned capacity, the *arachne* is shown to account for the so-called Platonic solution to the problem of the two mean proportionals and also for an Eudoxan heuristic method involving the use of «curved lines», mentioned by Eutocius. These have the same function as the auxiliary parameter (n) in the algebraic solution to the problem of combined spherical motions. Finally, the arachne, being also a (concrete) model of the geocentric system (just as Plato's world-soul is its metaphorical model), illustrates in cross section the tropics, equator, ecliptic, meridian, and the equinoctial points.

Certain pivotal features of the *arachne*, however, call forth further study. These are the possibility of changing the point of view while using the instrument, the preference of ratios of relative primes (which may be simultaneously odd) for ratios of composite integers, the optical quality of the solutions obtained, and the dynamic character of the heuristic methods discovered.

(I) The possibility of changing the view-point, not only in the sense of making use of the angle at circumference instead of the angle at the centre, but even more generally, permits us to look at the geocentric system through its model. Observations thus may be correlated to any of the great celestial circles and a given horizon simultaneously, and measurements made by means of the scales engraved in the circular mirror plate and in the measuring unit. Moreover, by means of the plumb-string dropped from the inclined meridian ring (the instrument in that case being supported by the circular plate), the plane projections of these great celestial circles can be obtained and even the *hippopede* may be produced in plane projection. This points the way to stereographic projection [11], and ultimately to the Alexandrian astrolabe. Finally, the change of view-point is exhibited in the connection between acute angles expressed in terms of tangents of their halves, harmonious points with given ratios of division, ratios of periods in terms of ratios of relative primes, and even musical intervals. That is to say, we discern a conscious attempt at universal scientific theories. Perhaps the most interesting detail is the use of the basic trigonometrical functions (the nomenclature being taken of course from the *gnomon*) anticipating Hipparchus's work.

(II) The preference of relatively prime numbers (these are met already in a preserved proof by Archytas) may be the most far-reaching idea connected with the *arachne*. When combined with the trigonometrical functions, they pave the way for the division of the circle. This was perhaps first achieved in the cases sin (circle: 12) $= 1:2$ and tan (circle : 8) $= 1:1$, and the former division may have been further developed on the analogy of the Egyptian calendar. According to the tradition, of course, the degrees were obtained from Mesopotamia in the second century B.C. (cf. Attalus Rhod. ap. Hipp. *In Arat.* 2.1.5 ; Böker, «Berichte u. Verh. der Sächs. Akad. der Wiss.». 99, 1952, H. 5, p. 52 ff, tried to show that Eudoxus's pupil Callippus invented the system). But on the one hand the method of exhaustion, perfected by Eudoxus, implies the idea of the division of the circle by means of regular polygons inscribed and circumscribing the circle, and on the other hand this is suggested also by the division of the longer side (lenght 360 units which may represent days) of the Platonic right-angled triangle giving the obliquity of the ecliptic by Plato's harmonic intervals [9].

Another implication of the relative prime numbers is their (heuristic) use in numerical analysis preparatory for axiomatic geometrical analysis. We have met ratios of relative primes in the solution to Eudoxus's problem of the combined spherical motions and in the ratios of division in harmonious points. But they are met also in the Pythagorean method of obtaining approximations to surds, which can be generalized so as to apply to square roots of all integers [12]. This strongly suggests that the main idea underlying the use of relative primes is the constructivity of numbers. We can follow this development from Pythagorean musical intervals to their oblong numbers $(n+1)n = 2+4+\ldots+2n$, and to the s u p e r p a r t i c u l a r e s $(n+1):n$ (as in Archytas), not to speak the Pythagorean analyses in terms of «the odd and the even»[9]. Further philosophical applications may be expected in

9. A fragment of Philolaus (Stob. *Ecl.* 1. 21. 7E) says that «number is of two special

(Xenocrates's?) «indivisible lines» and (Plato's?) «ideal numbers» [6] providing, perhaps, an analogy for Plato's semantical concepts of ὄνομα and ῥῆμα. In short, the relatively prime numbers seems to be a sub-set of positive integers having several desirable properties in heuristical methods, and lending themeslves readily to various models in philosophy.

(III) The optical character of all manipulations of the *arachne* is partly due to the fact that several central problems of ancient mathematics can be demostrated and solved by it, and partly to the fact that celestial objects are seen as reflections in the circular mirror plate (*ἔνοπτρον*). The *Enoptrika*, the title of a lost work of Philipp of Opus, may suggest that a school of «optical geometers» was active in Plato's and Eudoxus's time. And of course Plato himself compares the receptacle-space to a mirror in the *Timaeus*, which may account for the passages on mirrors and the mechanism of sight in the dialogue [5]. Moreover, it is well known that also Eudoxus wrote a work called the *Enoptron*. The mirror ensures that *reality* is depicted.

We have already mentioned the possibilities of the plane projections of the *hippopede* and the great celestial circles on the *enoptron*, the latter ones appearing as a family of arcs of circles through the points of support of the *enoptron* and corresponding to the visible parts of the great circles above the horizon. By their means several focal observations can be made. But the *arachne* also provides an instrumental solution to central problems of ancient mathematics other than so far discussed. For instance, Eudoxus's problem of the combined spherical motions can be divided into two parts (usually called, after Neugebauer, «the normal forms») which can be solved simultaneously. One concerns the determination of a rectangle with a given area by means of the measuring unit of the instrument, another the preserving of the sum or difference of the sides of the rectangle fixed. Especially, in the case where the auxiliary parameter $n=1$, the «curved line» known from the extraction of all cubic roots ($x^2 : y^2 = x^2 + y^2$) accounts for the second condition. The measuring unit ensures correspondence between reasoning and reality.

Part of the task consists of keeping the areas of the changing rectangles unaltered. In so doing, an apex of the rectangles, produced by the measuring unit, draws a hyperbola. But one of these rectangles is a square. Hence the

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kinds, odd and even, with a third, even-odd, arising from a mixture of the two; and of each kind there are many forms». This mixture may have been referred to in Plato's blending of the strange ingredients of the «material» of the world-soul in the *Timaeus*, 35a. The distinctions between prime, relatively prime and composite integers, and the «odds» and «evens» generating Pythagoren triples, suggest themselves.

arachne can be used in finding the *dynamis,* the quadratic value of a rectangle [12]. Moreover, even the problem known as the «application of the area» can be solved, and in this solution a parabola will be produced. And, indeed, we are told that Eudoxus's pupil Meneachmus invented the «conic sections». More likely, though, the use of the *arachne* suggested the hyperbola and the parabola without any cones. But of course the circular mirror plate of the instrument can also be used to demonstrate the circular section of various cones, the right cone among them.

The upshot of all these «optical solutions» and visible demonstrations, in the last analysis, may be epistemological[10]. As Arpád Szabó has shown, the very terms for proof in Greek mathematics derive from «showing» an- «demonstration», and we meet this older idea of proof applied to pedagogics in Plato (e.g. in the *Meno*). In Eudoxus, the optical resources are presumably utilized in the first place in the heuristics. But also the «eye-witness quality» of the Greek concept of knowledge should be borne in mind. Perhaps the *arachne* and the world-soul were not primarily conceived of as instruments or machines, but rather as m o d e l s. If so, they provided models both for the geocentric system and for epistemology in general [5,6]. Eudoxus's *arachne* seems to be the mundune counter part of Plato's cosmic computer, the world-soul.

(IV) Finally, the dynamic character of all these approximative methods manifest in the use of the *arachne,* akin to the Pythagorean methods of approximation to surds and their generalizations [12], point towards the «less respectable» tradition of the ancient mathematics represented by Archimedes, Heron and Diophantus, although their beginning may be seen in the finding of the *dynamis,* the quadratic value of a rectangle. If so, they motivate a re-examination of the modern evaluation of the relation between the axiomatic and instrumental methods in Greek mathematics. For these approximative instrumental methods induce a dynamic world-view which stands for the best inventive powers of the Greeks an their exact science.

T h e E c l i p s e a n d t h e C o r o n a.

The accuracy of Eudoxus's observations was eclipsed by Hipparchus's, his «curved lines» by Menaechmus's discovery of the curves we call the

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10. This would not lack an antecedent, for in a fragment Philolaus says much the same of the *gnomon*: «number makes all things knowable and mutually agreeing in the way characteristic of the *gnomon*»; Boeckh, *Philolaos des Pythagoreers Lehren,* pp. 141, 144. In Plato, again, in the fully developed doctrine of the receptacle space, mirrors have a similar epistemological rôle, and the receptacle itself is compared to a mirror. Perhaps the diagram in [5] will illustrate the case.

conic sections, his methods of invention by Archimedes's kindred methods, and his *arachne* by the astrolabe of the Alexandrian astronomers. But owing to these lost achievements' solid foundation on mathematical principles, we still discern their corona in Eudoxus's extant mathematical, astronomical and geographical results. It is from these results that our reconstruction of Eudoxus's methods and observational techniques has been built.

Hence our reconstruction of the *arachne* is a tribute to Eudoxus's *praxis*, deeply ingrained in theory yet capable of giving rise to a concrete, instrumental aid to invention and observation. Eudoxus's *Protophysik* is oriented towards geometry and his logic of discovery towards approximative methods. On the other hand, his observations are theory-informed and the most characteristic feature of his geometry and methods of proof (including the method of exhaustion) is their orientation towards motion. True, we could say that one function of the *arachne* alone, its use as a computing machine in the extraction of approximations to square and cubic roots, admits of a fairly safe dating. The other functions discussed belong to the *desiderata*. They would have been useful in Eudoxus's observations, but apart from such remarks as those of Plutarch, Eutocius, Hipparchus and Vitruvius, we know little about his instruments and techniques of observation. Yet, and this is the hallmark of the professional, of the «man of science if there ever was one», given one and the same instrument and one and the same method of computation, the expert is bound to investigate the whole compass of their applicability and draw the most immediate inferences from their use.

We may rest assured, therefore, that a mathematician of Eudoxus's competence investigated what else could be achieved by means of the same instrument that served so excellently in obtaining a practical solution to the problem of the two mean proportionals. Contrariwise, we can safely assume that an astronomer of Eudoxus's calibre also investigated other aspects than the observational of his astronomical instruments. For the very praxis of instruments and tools may teach the hand to continue the work of the intellect, presupposing that the instruments and tools have been constructed on sound, prolific theories.

Now we must emphasize that our reconstruction, too, is based on theoretical considerations of Eudoxus's work. These in turn are based on a series of studies in Plato's *Timaeus*, referred to in [1-12]. But the further we have proceeded, the more we have also gained feed-back that reinforces the previous, at times tentative results. Take for instance the philosophical and philological study of Plato's *agalma*. Starting from Platonic premisses it pointed out the remarkable role of a rotating model ascribed to the

World-Soul in the *Timaeus* It is a model that fully corresponds to physical reality at given times only. On the other hand, the reconstruction of Eudoxus's methods of computation in astronomy [8,9], being a study in the history of the exact sciences and starting from Eudoxan premisses, led to exactly the same concept of a model. And in the structure of the *arachne* we have a concrete manifestation of this concept, just as the usages of this instrument reflect not only Eudoxus's method of exhaustion and the Pythagorean approximative method, but finally also Plato's view on language (as described in E. Maula, *On the Semantics of Time in Plato's Timaeus*, «Acta Acad. Aboensis», Ser. B, Tom. 169,1, 1970). That these results, obtained from fairly divergent premisses, tend to converge towards a new synthesis of the ancient scientific world-view, cannot be mere coincidence. For just as it was the case with our reconstruction of Eudoxus's cosmological system, there are too many exact parameter values deduced from the relatively few known values by one and the same method and constituting a sound logical system, to be explained away by mere chance.

Nay, the *arachne* reconstructed here stands at the watershed between theory and praxis, between proof and heuristics, between the axiomatic method and the methods of invention, between geometrical and numerical analysis, between rational and irrational numbers, and between the static and dynamic world-views. But owing to the depth of Eudoxus's insight, these seemingly opposite traditions unite. The approximative and the exact, the practical and the theoretical, turn out to be two aspects of the same synthesis, which is the bearing force behind Eudoxus's lasting contribution to the development of mathematical analysis. The *arachne* is a «living statue» just as Plato's *agalma*, a paradigm of radiant *ingenium* commemorating the interdisciplinary approach of Eudoxus, attested also by his inquiries into the interrelation of astronomy and music (Theon Smyrn., p. 61 Hiller). For the *arachne*, in contradistinction to the ruler and compasses which already had undergone a metamorphosis into idealized symbolic instruments, has preserved the heuristical pregnancy of a mathematical tool invented by a *homo ludens*. However, the *enoptron* reflecting the physical reality and the measuring unit's manipulations ensuring that the laws of thought correspond to physical laws, the *arachne* is also a powerful answer to Zeno's paradoxes.

Finally, despite the fact that the *arachne* is a theoretical reconstruction, we would not be over much surprised even if a concrete remnant of the instrument (perhaps the *enoptron*, possibly classified as «a mirror with mathematical and astronomical decorations») were rediscovered. Such a finding, indeed, would contribute to our better understanding of the beginnings of the geometry of motion and the geometrical treatment of time. For the *arachne* is, of course, a plane projection of the pivotal features of the ever rotating spheres*.

* Note added to the proof: In August, 1976, on a voyage to Cnidus, we received a miraculous message from the Cnidian Aphrodite through Professor Iris Cornelia Love, American archaeologist who has excavated a «temple of Aphrodite» exhibiting several unique features, and in its vicinity, other scientific instruments. This «temple» in fact is an accurate copy, sixty times magnified, of the *arachne* —even the *gnomon* at the centre was there, excavated *in situ*. And moreover, the auxiliary trank, the *dioptra*, of bronze, had been found— tentatively classified as «a key to the gate of the temple». These exciting finds, however, deserve a new paper. One first report is [13]. According to Professor Love, these finds and the whole town-plan very likely are due to Eudoxus.

ΤΟ ΠΡΩΤΟ ΓΩΝΙΟΜΕΤΡΟ ΓΙΑ ΤΗ ΜΕΤΡΗΣΗ ΤΩΝ ΟΥΡΑΝΙΩΝ ΣΦΑΙΡΩΝ. Η «ΑΡΑΧΝΗ» ΤΟΥ ΕΥΔΟΞΟΥ

Περίληψη.

Ἡ μελέτη αὐτὴ ἀποτελεῖ ἀναθεωρημένη καὶ γενικευμένη παρουσίαση τῆς ἀνακοινώσεως ποὺ ἔγινε στὸ «Διεθνὲς Συνέδριο γιὰ τὴν ἱστορία τῆς Μετρήσεως καὶ τὸ ρόλο τῶν κανόνων της στὸν πολιτισμὸ» (Βουδαπέστη, Ἀπρίλιος 1976). Τὸ γενικὸ θέμα τοῦ Συνεδρίου περιώρισε τὴν ἀνακοίνωση στὴν ἀρχαιότατη ἱστορία τῆς μετρήσεως γωνιῶν. Οἱ φιλοσοφικὲς ἐπιπτώσεις τῆς τεχνικῆς τῶν μετρήσεων, ποὺ ἀναλύονται ἐδῶ, δὲν συζητήθηκαν στὸ συνέδριο. Στὴν περίληψη αὐτὴ συγκεφαλαιώνομε τοὺς συλλογισμούς, ποὺ μᾶς ὡδήγησαν στὴν ἀνακατασκευὴ τῆς «ἀράχνης», τοῦ κυρίου ὀργάνου μετρήσεως γωνιῶν τοῦ Εὐδόξου. Συμπερασματικὰ προσπαθοῦμε νὰ δείξωμε πῶς ἡ μέθοδος ποὺ χρησιμοποιεῖται στὸ ὄργανο αὐτὸ ἐγκαινιάζει μιὰ νέα δυναμικὴ ἀντίληψη τοῦ κόσμου.

Ὡς ἀφετηρία στὴν ἔρευνά μας πήραμε τὴ «μεγάλη ἁρμονία» τοῦ Πλάτωνος στὸν *Τίμαιο*, μὲ βάση τὴν ὑπόθεση ὅτι τὸ οἰκοδόμημα τῆς ψυχῆς τοῦ κόσμου συναρμολογήθηκε εἴτε μὲ πλαίσιο ἀναφορᾶς τὸ κοσμολογικὸ σύστημα τοῦ Εὐδόξου εἴτε προωθώντας τὸ σύστημα αὐτό. Στὴ συνέχεια ἀνακαλύψαμε ὅτι ἡ τιμὴ τῆς λοξώσεως τῆς ἐκλειπτικῆς, ποὺ χρησιμοποίησε ὁ Πλάτων καὶ πιθανῶς ὁ Εὔδοξος, εἶναι κάπως μικρότερη ἀπὸ τὴν πραγματικὴ τιμή. Ἡ διαπίστωση, ὅτι ἡ τιμὴ τῆς λοξώσεως τοῦ Πλάτωνος εἶχε πραγματικὰ χρησιμοποιηθῆ καὶ ἀπὸ τὸν Εὔδοξο, ἐνισχύεται ἀπὸ τὴ μελέτη τῆς μεθόδου, ποὺ χρησιμοποιεῖ γιὰ νὰ προσδιορίζη τὸ γεωγραφικὸ πλάτος.

Ἡ χρησιμοποίηση τοῦ νέου αὐτοῦ στοιχείου ἐπέτρεψε τὴν ἀνασύνθεση

τῆς μεθόδου ὑπολογισμοῦ τοῦ Εὐδόξου στὶς πλανητικὲς θεωρίες του κι ἀκόμη στὶς μεθόδους ἀναλύσεως καὶ συνθέσεως καθὼς καὶ σ' ἕνα οὐσιῶδες χαρακτηριστικὸ τῆς εὑρετικῆς του μεθόδου. Τὸ σύστημα τοῦ Εὐδόξου καὶ ἡ ψυχὴ τοῦ κόσμου, τὴν ὁποία ὁ Πλάτων συγκρίνει μὲ *ἄγαλμα*, παρουσιάζουν τὰ ἴδια χαρακτηριστικὰ εἰς τὰ πρότυπά τους, ποὺ ἀνταποκρίνονται ἀπόλυτα στὴ φυσικὴ πραγματικότητα, μόνον ὅμως σὲ δεδομένους χρόνους. Πέρα ἀπὸ τὴν ἀσυνέχεια αὐτή, τὸ σύστημα τοῦ Εὐδόξου, ποὺ ἀνασυνθέσαμε, ἐξηγεῖ πλήρως μὲ ἀφετηρία τὶς γνωστὲς περιόδους τοῦ Εὐδόξου περισσότερες ἀπὸ ἑκατὸ ἄλλες κύριες παραμέτρους τῆς κοσμολογίας του.

Γνωρίζοντας, λοιπόν, ὅτι στὴ λύση τοῦ προβλήματος τῶν συνδυασμένων σφαιρικῶν κινήσεων ὁ Εὔδοξος ἐπέτυχε ὀξεῖες γωνίες μετρούμενες μὲ τὴν ἐφαπτομένη μισῶν γωνιῶν, ἀνακατασκευάσαμε τὴν «ἀράχνη», ποὺ μετρᾶ τὶς γωνίες κατ' αὐτὸ τὸν τρόπο. Ἡ «ἀράχνη» μπορεῖ ἀκόμη νὰ χρησιμοποιηθῆ γιὰ τὸν προσδιορισμὸ τοῦ γεωγραφικοῦ πλάτους, σύμφωνα μὲ τὴ μέθοδο τοῦ Εὐδόξου, καὶ γιὰ τὴν ἐξαγωγὴ τετραγωνικῶν καὶ κυβικῶν ριζῶν. Σὲ μιὰ προσεκτικότερη ἀνάλυση τῆς δυνατότητας ποὺ ἀναφέραμε τελευταῖα, ἀποδεικνύεται ὅτι ἡ ἀράχνη ἐξηγεῖ τὴν ἐπονομαζόμενη Πλατωνικὴ λύση στὸ πρόβλημα τῶν δύο μέσων ἀναλόγων καὶ μιὰ εὑρετικὴ μέθοδο τοῦ Εὐδόξου, ποὺ συνεπάγεται τὴ χρήση τῶν «καμπύλων γραμμῶν» καὶ ποὺ ἀναφέρει ὁ Εὐτόκιος. Αὐτὲς λειτουργοῦν ὅπως ἡ βοηθητικὴ παράμετρος στὴν ἀλγεβρικὴ λύση τοῦ προβλήματος τῶν συνδυασμένων σφαιρικῶν κινήσεων. Τὸ συμπέρασμα εἶναι ὅτι ἡ «ἀράχνη» μὲ τὸ νὰ εἶναι ἕνα συγκεκριμένο πρότυπο τοῦ γεωμετρικοῦ συστήματος (ὅπως ἀκριβῶς ἡ ψυχὴ τοῦ κόσμου τοῦ Πλάτωνος εἶναι τὸ μεταφορικὸ πρότυπό της) εἰκονογραφεῖ τοὺς τροπικούς, τὸν ἰσημερινό, τὴν ἐκλειπτική, τὸν μεσημβρινὸ καὶ τὰ ἰσημερινὰ σημεῖα.

Ὁ δυναμικὸς χαρακτήρας, τέλος, ὅλων αὐτῶν τῶν μεθόδων προσεγγίσεως, ποὺ εἶναι φανερὲς στὴ χρήση τῆς ἀράχνης, συγγενικὲς μὲ τὶς πυθαγόρειες μεθόδους προσεγγίσεως τῶν ἀσυμμέτρων ἀριθμῶν καὶ τῶν γενικεύσεών της, στρέφουν τὴν προσοχὴ πρὸς τὴ λιγότερο ὀρθόδοξη παράδοση τῶν ἀρχαίων μαθηματικῶν, ποὺ ἀντιπροσωπεύθηκαν ἀπὸ τὸν Ἀρχιμήδη, τὸν Ἥρωνα καὶ τὸν Διόφαντο, ἂν καὶ ἡ ἀρχή τους μπορεῖ νὰ ἀνιχνευθῆ στὴν εὕρεση τῆς δυνάμεως, τὴν δευτεροβάθμια τιμὴ ἑνὸς ὀρθογωνίου. Ἂν εἶναι ἔτσι, ὅλα αὐτὰ παρακινοῦν γιὰ μιὰ ἐπανεξέταση τῆς νεώτερης ἐκτιμήσεως τῆς σχέσεως μεταξὺ τῆς ἀξιωματικῆς μεθόδου καὶ τῆς χρήσεως ὀργάνων στὰ ἑλληνικὰ Μαθηματικά. Γιατὶ αὐτὲς οἱ «ἐνόργανες» μέθοδοι προσεγγίσεως συνεπάγονται μιὰ δυναμικὴ ἀντίληψη τοῦ κόσμου, ποὺ ἐκφράζει τὶς πιὸ δημιουργικὲς δυνάμεις τῶν Ἑλλήνων στὴν θετικὴ ἐπιστήμη τους.

(Μετάφραση Μ. Δραγώνα - Μονάχου).